Representation theory notes

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1 Basics

We begin by working over a generic field F. (At a certain point, we restrict to $F = \mathbb{C}$.)

Defn: a representation of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Dimension of ρ is $\dim V$.

Defn: a G-homomorphism between G-reps V, W is a linear map $\phi : V \to W$ s.t. $\phi(gv) = g\phi(v)$. $\mathrm{Hom}_G(V, W)$ is the space of G-homs.

Defn: the G-reps V, W are equivalent/isomorphic if there exists a bijective G-hom $\phi : V \to W$.

Examples: trivial rep, regular rep $FG = \{\sum_g a_g e_g\}$ with G acting on itself by left-multiplication, permutation representation of S_n.

It is natural to ask whether we can classify reps. The relevant building blocks are the following.

Defn: an irreducible rep V is a rep with no nontrivial G-stable subspace. (A rep with a G-stable subspace is reducible.)

Defn: a rep V is decomposable if $V = U \oplus W$ for nontrivial subreps U, W. (A rep that cannot be written in this fashion is indecomposable.)

Maschke's Thm: if char$F$ does not divide $|G|$, then V reducible implies V decomposable. In other words, if U is a subrep, then there exists a subrep W s.t. $V = U \oplus W$.

Pf: construct W as an orthogonal complement with respect to a G-invariant inner product. Construct this inner product by averaging over G. Alternatively, construct a G-equivariant projection map $p : V \to U$ and set $W = \ker p$.

Remark: this fails when char$F$ divides $|G|$. This case leads to modular representation theory.

Cor (complete reducibility): any rep can be written as the direct sum of irreps.

Schur's Lemma: let V, W be G-irreps.

1. every G-hom $\phi : V \to W$ is either 0 or invertible.

2. if F is algebraically closed, then every G-hom $\phi : V \to W$ is of the form $\phi = \lambda \mathrm{id}$.

Pf: $\ker\phi, \mathrm{im}\phi$ are subreps of the irreps V, W, respectively.

In other words, Schur's Lemma states that the G-equivariant hom space $\mathrm{Hom}_G(V, W)$ has dimension $\mathbf{1}_{V \simeq W}$.

We now set $F = \mathbb{C}$ and $|G| < \infty$ for the remainder of the course.

Abelian groups have particularly simple irreps.

Prop: let V be G-irrep. Then there exists $\lambda : Z(G) \to \mathbb{C}^\times$, the central character, s.t. $z \cdot v = \lambda(z)v$ for all $z \in Z(G)$. In other words, restriction to $Z(G)$ lands in scalar matrices.

Pf: note that $\phi_z : V \to V$ given by $v \mapsto z \cdot v$ is a G-hom, since $z \in Z(G)$ implies $\phi_z(gv) = g\phi_z(v)$ for all g, v. By Schur, $\phi_z = \lambda \mathrm{id}$, as desired.

Thm: if G finite and abelian, then every irrep is 1 dimensional. Further, there are $|G|$ irreps.

We now consider uniqueness of decomposition. We begin with a study of the G-hom subspace.

Lem: if V, W are G-irreps, then $\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & V \simeq W \\ 0 & \text{else} \end{cases}$.

Lem: let V, V_1, V_2 be G-irreps. Then we have $\mathrm{Hom}_G(V, V_1 \oplus V_2) \simeq \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$ and likewise in the first entry.

Cor: let V be a G-rep with $V \simeq V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}$ with V_i distinct irreps. Then $n_i = \dim \mathrm{Hom}_G(V_i, V)$.

Thm (isotypical decomposition): let V be a G-rep with $V \simeq V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ $(*)$. Then

1. the integers n_i are uniquely determined.

2. the image of $V_i^{n_i}$ in V under any isomorphism in $(*)$ is independent of the choice of isomorphism; this is the V_i-isotypic component of V.

Remark: this parallels the decomposition of V into eigenspaces of a given endomorphism T.

Therefore $V = W_1 \oplus \cdots \oplus W_k$ for isotypic components W_i (not just isomorphism; equality!).

2 Character theory

Defn: the character of a rep $\rho : G \to \mathrm{GL}(V)$ is the map $\chi_V : G \to \mathbb{C}$ given by $\chi_V(g) = \mathrm{tr}(\rho(g))$.

Defn: let $\mathcal{C}(G)$ denote the space of class functions on G.

Note that $\dim \mathcal{C}(G) = |\{\text{conjugacy classes of G}\}|$.

Prop: let V be G-rep. Then

1. $\chi(1) = \dim V$
2. χ is a class function
3. $\chi(g^{-1}) = \overline{\chi(g)}$
4. $\chi_{V \oplus W} = \chi_V + \chi_W$

Defn: given $f, f' \in \mathcal{C}(G)$, define the Hermitian inner product $\langle f, f' \rangle = \frac{1}{|G|} \sum_g f(g)\overline{f'(g)}$.

The following is the main theorem of character theory.

Thm: the irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$.

We begin with some consequences.

Prop: V is irrep iff $\langle \chi, \chi \rangle = 1$.

Note that for a permutation rep, $\chi(g)$ equals the number of fixed points of $g\cdot$. We can see this by writing the permutation matrices.

Lem: $\chi_{reg}(g) = \begin{cases} |G| & g = 1 \\ 0 & \text{else} \end{cases}$.

Thm: if $d_i = \dim V_i$ and V_i are the irreps of G, then $\sum d_i^2 = |G|$.

Pf: compute $\langle \chi_{reg}, \chi_{reg} \rangle$ in two ways.

Character tables list the irreps in the rows and conjugacy classes in the columns.

From the main theorem, the rows form an orthonormal basis. In fact we have the following.

Prop (column orthogonality): $\sum_\chi \chi(g_i)\overline{\chi(g_i)} = \begin{cases} |C_G(g_i)| & i = j \\ 0 & \text{else} \end{cases}$.

Lem:

1. χ irreducible character implies $\overline{\chi}$ irreducible character.
2. χ irreducible character and ε 1-dimensional character implies $\varepsilon\chi$ is irreducible character.

We return to the proof of the main theorem.

Prop: $\langle \chi_V, 1 \rangle = \langle 1, \chi_V \rangle = \dim V^G$, where $V^G = \{v | gv = v \forall g\}$ is the G-invariant subspace of V where G acts trivially.

Pf: let $\phi = \frac{1}{|G|} | \sum_{g \in G} g \in \mathrm{End}(V)$. Then $\phi : V \to V$ is a projector onto V^G. The kernel of a projector equals the dimension of the image. Hence $\dim V^G = \mathrm{tr}\phi = \frac{1}{|G|} \sum_{g \in G} \chi(g) = \langle \chi, 1 \rangle$, as claimed.

In order to continue, we need to define the hom space rep $\mathrm{Hom}(V, W)$. The rep is defined by the G-action $g \cdot \phi(v) = g \cdot \phi(g^{-1} v)$ for all $\phi \in \mathrm{Hom}(V, W), g \in G$. We can think of $\mathrm{Hom}(V, W)$ as a space of matrices, and the action "conjugates" by g in some sense. In particular, $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$. When $W = \mathbb{C}$, we find the dual rep V^* consisting of row vectors with the action $g \cdot \phi(v) = \phi(g^{-1} v)$; ϕ gets sent to the row vector $\phi \cdot g^{-1}$.

Lem: $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \chi_W$.

Prop: for any G-reps V, W, we have $\langle \chi_V, \chi_W \rangle = \langle \chi_W, \chi_V \rangle = \dim \mathrm{Hom}_G(V, W)$.

Pf: apply the previous Prop to $V \mapsto \mathrm{Hom}(V, W)$. Then $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \chi_W$. Also, $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$, which has dimension $\mathbf{1}_{V \simeq W}$. This implies the result.

For completeness, go by contradiction and construct a special G-hom ϕ_V.

We now recall a few concepts from linear algebra.

Defn: given V, W with bases $\{e_i\}, \{f_j\}$, the tensor product $V \otimes W$ is the vector space spanned by the formal symbols $\{e_i \otimes f_j\}$.

Note that $\dim V \otimes W = \dim V \dim W$.

Prop (universal property): $\mathrm{Hom}(V \otimes W, U) \simeq \{\text{bilinear maps } V \times W \to U\}$.

Lem: there is a natural iso $V^* \otimes W \simeq \mathrm{Hom}(V, W)$.

Note that S_n acts on $V^{\otimes n}$ in the following way: $\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. The permutation σ permutes the components of the pure tensor. For instance, (123) sends $v_1 \otimes v_2 \otimes v_3$ to $v_3 \otimes v_1 \otimes v_2$, since each vector gets shifted one to the right. Note the inverse!

Defn: the symmetric power $\mathrm{Sym}^n V$ as $\{x \in V^{\otimes n} | \sigma(x) = x\}$.

Defn: the exterior power $\bigwedge^n V = \{x \in V^{\otimes n} | \sigma(x) = -x\}$.

Given a basis $\{e_i\}$ for V, bases for $\mathrm{Sym}^n V$ or $\bigwedge^n V$ can be written $\{e_{i_1} \cdot \ldots \cdot e_{i_n}\}$ and $\{e_{i_1} \wedge \ldots \wedge e_{i_n}\}$.

Add tensor products of reps...

3 Algebras

Defn: a unital associative algebra is a vector space A over F with a bilinear map $A \times A \to A$, $(a, b) \mapsto a \cdot b$ such that

1. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. there exists an identity $1 \in A$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in A$.

Simply put, an algebra is a vector space which also has multiplication.

Examples:

1. $A = \mathrm{Mat}_n(\mathbb{C})$. (This is the example to keep in mind when thinking of algebras.)
2. $A = \mathrm{End}(V)$. (This is the same as the previous example. This is *different* from the group of invertible matrices $\mathrm{GL}(V)$, which is *not* a vector space.)
3. $A = \mathbb{C}[x]$.

Defn:

1. given algebras A, B, a linear map $f : A \to B$ is an algebra hom if $f(ab) = f(a)f(b)$ and $f(1) = 1$.
2. a rep of an algebra A is an algebra hom $A \to \mathrm{End}(V)$.
3. a left A-module is a vector space V with left multiplication by A: $1 \cdot v = v$ and $a(bv) = abv$.
4. a linear map $f : V \to W$ is an A-module hom if $f(am) = af(m)$.

Intuitively, a rep of an algebra A is an association of a matrix to each element $a \in A$. These are not necessarily invertible matrices; for instance, $0 \in A$ is sent to the zero matrix!

Since a rep of an algebra defines left multiplication by A, we see that {left A-modules} $\overset{1:1}{\leftrightarrow}$ {reps of A}. Representation theory of algebras is module theory over algebras.

It is easy to get confused by the above terminology. Think of an algebra A as the analogue of a group, and think of reps as A-modules.

Examples:

1. every algebra A has the natural rep $V = A$, where A acts by left multiplication.
2. $A = \mathrm{Mat}_n(\mathbb{C})$ has a natural rep $V = \mathbb{C}^n$, where A acts by matrix multiplication on column vectors.

Add...