ORF527 review

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0.1 Remember

When in doubt: apply Itô's formula!

0.2 Shorthand

[STS = suffices to show, wrt = with respect to, mb = measurable, ac = absolutely continuous, SP = stochastic process, pm = progressively measurable, BM = Brownian motion, MG = martingale, loc MG = local martingale, BV = bounded variations, QV = quadratic variation, CV = cross-variation, SI = stochastic integral, IC = initial condition, DDS = Dambis–Dubins–Schwarz]

0.3 Stochastic basis

- Continuous time SPs: same old. Always working on filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying "usual conditions" (right-continuous and complete).
- $\mathbb{F}$-adapted: same old.
- $\mathbb{F}$-pm: the SP X is pm if for all $t \geqslant 0$, the mapping $X : ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. (pm implies mb and adapted.)
- Stoppping time/stopped σ-algebra: same old.
- Optional stopping: M continuous MG, τ, σ two *bounded* stopping times with $\sigma \leqslant \tau$ $\mathbb{P}$-as. Then MG property holds for stopping times: $E[M_\tau | \mathcal{F}_\sigma] = M_\sigma$ $\mathbb{P}$-as. (In particular, $E[M_\tau] = M_0$.) (Many other versions exist.)
- $\mathbb{F}$-BM: an $\mathbb{F}$-adapted SP s.t. $W_0 = 0$, independent increments, and gaussian increments.
- MG: $\mathbb{F}$-adapted, integrable, and MG property (all three!).
- Loc MG: if there exists sequence of stopping times (τ_n) increasing to ∞ $\mathbb{P}$-as s.t. X^{τ_n} is MG for all n. (Relevant when X fails the integrability condition of a MG.)

- BV on $[0,t]$: when $\sup_{\Pi=(t_k)} \sum |\Delta X_{t_k}| < \infty$.
- QV on $[0,t]$: same, but with increments squared. Denoted $\langle X \rangle$.
- Doob–Meyer v1: if $M \in \mathcal{M}_c^2$ (so that $M_0 = 0$), then there exists unique continuous nondecreasing process $[M]$ s.t. $[M]_0 = 0$ and $M^2 - [M]$ is a MG. In fact, $[M] = \langle M \rangle$.
- Doob–Meyer v2: if $M, N \in \mathcal{M}_c^2$, then there exists unique continuous predictable BV process $\langle M, N \rangle$ s.t. $\langle M, N \rangle_0 = 0$ and $MN - \langle M, N \rangle$ is a MG. This is denoted CV. (Doob–Meyer v1 implies parallelogram law: $\langle M, N \rangle = \frac{1}{4}(\langle M+N \rangle - \langle M-N \rangle)$.)

(bounded loc MG is genuine MG, bracket constantly zero $\implies$ constant, independent $\implies$ zero CV)

0.4 Stochastic integral

Goal: integrate a "nice" SP X wrt a MG M to define a process called the SI. (More generally, we want to integrate wrt a loc MG.)

(In the discrete setting, the predictable SP X was our holdings of a stock, and the MG M was the stock price. It was clear that the discrete MG transform $I(X)$ should be a MG, since there's no way to make money off an unpredictable MG!)

For simplicity, we construct the SI when M is a MG and when X lies in the following space.

Defn: given $M \in \mathcal{M}_c^2$, the space $\mathcal{L}_M^*$ is defined as

$$\mathcal{L}_M^* := \{X : \mathbb{F}\text{-pm}, E[\int_0^t X_s^2 d\langle M \rangle_s] < \infty \forall t \geqslant 0 \ \mathbb{P}\text{-as}\} \tag{1}$$

Step 1: $X \in \mathcal{L}_0$, simple processes. Explicitly, a piecewise constant process wrt a fixed partition (t_i): $X_t = \xi_0 \mathbf{1}_0(t) + \sum_{i=0}^{\infty} \xi_i \mathbf{1}_{(t_i, t_{i+1}]}(t)$ such that ξ_i is $\mathcal{F}_{t_i}$-mb and (ξ_i) uniformly bounded by some K. Clearly $\mathcal{L}_0$ is a subspace of $\mathcal{L}_M^*$.

We make the obvious definition $I_t(X) := \sum_{i=0}^{\infty} \xi_i (M_{t \wedge t_{i+1}} - M_{t \wedge t_i})$. (Note only finite many terms are nonzero.)

Prop (properties of SI): if $M \in \mathcal{M}_c^2$ and $X \in \mathcal{L}_0$, then $I_0 = 0$, I is a (square-integrable) MG, we have $E[|I_t|^2] = E[\int_0^t |X_s|^2 d\langle M \rangle_s]$, and the map $I : \mathcal{L}_0 \to \mathcal{M}_c^2$ is linear in X.

[To check the third, prove the conditional version: $E[|I_t - I_s|^2 | \mathcal{F}_s] = E[\int_s^t |X_u|^2 d\langle M \rangle_u | \mathcal{F}_s]$. At some point use Doob–Meyer to show $E[(M_u - M_t)^2 | \mathcal{F}_s] = E[M_u^2 - M_t^2 | \mathcal{F}_s] = E[\langle M \rangle_u - \langle M \rangle_t | \mathcal{F}_s.]$

By the conditional version of the above and Doob–Meyer, we obtain (what we *really* call) the **Itô isometry**, without the expectation: $\langle I \rangle_t = \int_0^t |X_u|^2 d\langle M \rangle_u$. [Just check $I^2 - \langle I \rangle$ is a MG.]

Step 2: an approximation argument: $\mathcal{L}_0$ is dense in $\mathcal{L}_M^*$ wrt a certain distance d.

[Explicitly, $d(X,Y) := \sum_{n \geqslant 1} 2^{-n} \cdot 1 \wedge E[\int_0^n |X_s - Y_s|^2 d\langle M \rangle_s]$, which is the canonical way of patching together countably many L^2 spaces. Convergence in d is equivalent to each summand going to zero. In

the course of the argument, we reduce to the case that $X \in \mathcal{L}^*_M$ is continuous and bounded and $d\langle M\rangle$ is ac wrt Lebesgue. Then we may construct an approximating sequence of left Riemann sums X^i which converge pointwise to X, so that by DCT we have L^2 convergence on $[0,n]$ for each n, as desired.]

Step 3: extend SI to all of $\mathcal{L}^*_M$. Given $X \in \mathcal{L}^*_M$, choose $(X^i) \subseteq \mathcal{L}_0$ such that $X_i \to X$ wrt d. Using the Itô isometry, can show that for each t, $(I_t(X^i))_{i\geqslant 1}$ is a Cauchy sequence in $L^2(\Omega)$, hence by completeness the sequence converges. It is an easy fact that if a sequence of square-integrable MGs converges pointwise to a given SP, then the limit is itself a square-integrable MG. Thus there exists a limit $I(X) \in \mathcal{M}^2_c$, the SI of X wrt M.

[To check L^2 Cauchy sequence: note by Itô isometry that

$$E[|I^m_t - I^n_t|^2] = E[|\int_0^t (X^m_s - X^n_s)dM_s|^2] \tag{2}$$

$$\text{(Doob–Meyer)} = E[\langle \int_0^{\cdot} (X^m_s - X^n_s)dM_s\rangle_t] \tag{3}$$

$$= E[\int_0^t (X^m_s - X^n_s)^2 d\langle M\rangle_s] \tag{4}$$

$$\leqslant \epsilon \tag{5}$$

for all $m,n \geqslant N$, since (X^i) forms a Cauchy sequence wrt d.]

Claim all properties go through: if $M \in \mathcal{M}^2_c$ and $X \in \mathcal{L}^*_M$, then $I(X)$ is a genuine MG and the Itô isometry $\langle I\rangle_t = \int_0^t |X_s|^2 d\langle M\rangle_s$ holds.

General Itô isometry: if $M,N \in \mathcal{M}^2_c$, $X \in \mathcal{L}^*_M$, $Y \in \mathcal{L}^*_N$ (different MGs!), then

$$\langle \int_0^{\cdot} X_s dM_s, \int_0^{\cdot} Y_s dN_s\rangle_t = \int_0^{\cdot} X_s Y_s d\langle M,N\rangle_s \tag{6}$$

Rmk: the Itô isometry is very intuitive. QV accumulates moment by moment, and the QV of the SI increment $X_s dM_s$ is the QV of M at that instant amplified by the factor X_s; hence $X^2_s d\langle M\rangle_s$. It is called an isometry because the mapping $X \mapsto I(X)$ links two different norms, namely $\int X^2 d\langle M\rangle$ and $\langle I(X)\rangle$.

Rmk: if M is merely a *loc MG* and the $\mathbb{F}$-pm SP X satisfies the *weaker* integrability condition that

$$\int_0^{\infty} X^2_s d\langle M\rangle_s < \infty \tag{7}$$

$\mathbb{P}$-as, then we can still construct $I(X)$ to be a *loc MG*, but we *cannot* say that $I(X)$ is a genuine MG. ("SI preserves loc MGs.")

Rmk: since integrating wrt a loc MG makes sense and yields a loc MG, integrating wrt a semiMG ($dX_t = dM_t + dA_t$) makes sense as a sum of a SI and a LS integral.

0.5 Itô's formula

Defn: a continuous semiMG is a SP X such that $X_t = X_0 + M_t + A_t$, where X_0 is $\mathcal{F}_0$-mb, M is a continuous loc MG with $M_0 = 0$, and A is a continuous adapted BV SP. (We assume square integrable below.)

(Note for a continuous semiMG, $\langle X \rangle = \langle M \rangle$.)

Common class of semiMGs: Itô processes of the form $dX_t = b_t dt + \sigma_t dW_t$, where: X_0 is $\mathcal{F}_0$-mb, b, σ are $\mathbb{F}$-pm, W is a BM, $\int_0^t |b_s| ds < \infty$ for all t, and $\sigma \in \mathcal{L}^*_W$. Simply put, an Itô process is drift + BM. The log-price of a stock follows an Itô process. *(We're not just applying Itô to MGs!)*

Given a $\mathbb{R}$-valued continuous semiMG X and a "nice" $f : \mathbb{R} \to \mathbb{R}$, what is the formula for the SP $f(X)$?

Itô's formula (1-d): if X is continuous semiMG and $f : \mathbb{R} \to \mathbb{R}$ is C^2, then if $Y := f(X)$,

$$dY_t = f'(X_t) dX_t + \frac{1}{2} f''(X_t) d\langle X \rangle_t \tag{8}$$

Equivalently,

$$Y_t = f'(X_t) dA_t + \frac{1}{2} f''(X_t) d\langle M \rangle_t + f'(X_t) dM_t \tag{9}$$

(We can write everything in integral form; don't forget IC! The first two terms yield LS integrals and hence BV processes, while the last term is a SI. Apriori the last term only yields a loc MG, not a genuine MG! Y is again a continuous semiMG.)

Multi-d Itô's formula: let $X = (X^1, \ldots, X^d)$ be an $\mathbb{R}^d$-valued SP, where X^i is a continuous semiMG for all i. Let $f : \mathbb{R}^d \to \mathbb{R}$ be C^2. Then the $\mathbb{R}$-valued SP $Y = f(X)$ is given as

$$dY_t = \sum_i \partial_i f(X_t) dX^i_t + \frac{1}{2} \sum_{i,j} \partial_{ij} f(X_t) d\langle X^i, X^j \rangle_t \tag{10}$$

(We can write everything in integral form; don't forget IC! By expanding further, we see that Y is again a continuous semiMG.)

[Proof in 1-d: first, reduce to the bounded case using the localizing sequence

$$\tau_n = \{\text{any constituent process exceeds n in magnitude}\} \tag{11}$$

and sending $n \to \infty$. Assuming all processes are bounded by K, we partition $[0, t]$ into $(t_i)_{0 \leqslant i \leqslant n}$ and simply Taylor expand in each subinterval. (Not over all of $[0, t]$!) Using shorthand, this yields

$$f(X_t) - f(X_0) = \sum_i f'(X_i) \Delta X_i + \frac{1}{2} \sum_i f''(X_i) (\Delta X_i)^2 \tag{12}$$

The first sum splits into $\sum_i f'(X_i) \Delta A_i + \sum_i f'(X_i) \Delta M_i$. Since integrand is bounded, the first goes to LS integral. Since integrand is bounded simple process, the second goes in L^2 to SI. Now consider the second sum; by expanding $(\Delta X_i)^2 = (\Delta M_i)^2 + (\Delta A_i)^2 + 2 \Delta M_i \Delta A_i$, sup-bounding shows only first term survives; the rest are $o(\Delta t)$. Thus last term goes in L^2 to LS integral wrt $d\langle M \rangle$. Put together the terms.]

0.6 Martingale representation

How can we represent a generic continuous square integrable MG in terms of the universal object BM?

The key tool is the following surprising **characterization of BM**. It has a remarkably analytical proof.

Thm (Lévy characterization): suppose $M \in \mathcal{M}_c^2$ (with $M_0 = 0$) such that $\langle M \rangle_t = t$ for all $t \geqslant 0$. Then M is a BM (wrt the probability space on which we are working).

[STS increments are conditionally gaussian. Show that the characteristic function $\phi(t)$ of $M_t - M_s$ conditional on $\mathcal{F}_s$ is $\exp(-u^2(t-s)/2)$; minus sign! Apply (unconditional) Itô to $X_r = f(M_r)$, where $f(x) = \exp(iux)$ and $r \in [s,t]$. Take conditional expectation of $X_t - X_s$ and manipulate to get ODE for $\phi(t)$.]

DDS Thm (MG as time-changed BM): suppose $M \in \mathcal{M}_c^2$ (so that $M_0 = 0$) and $\langle M \rangle_t \to \infty$. Define the stopping time $\tau_s := \inf\{t \geqslant 0, \langle M \rangle_t \geqslant s\}$. Define the SP W given by $W_s := M_{\tau_s}$. Define the time-changed filtration $\mathbb{G}$ given by $\mathcal{G}_s := \mathcal{F}_{\tau_s}$. Then W is a BM wrt the filtration $\mathbb{G}$. Moreover, $M_t = W_{\langle M \rangle_t}$.

(Note the time-changed filtration! By Lévy, this is the *only* possible way to reparametrize M to obtain BM; W must accumulate QV at the correct rate. Apriori $\langle M \rangle$ is nondecreasing. But if $\langle M \rangle$ goes constant, i.e. if M is stopped somehow, then clearly we cannot hope to represent M as a BM.)

[Must check W is square integrable MG, $\langle W \rangle_t = t$, and W is continuous. First two are easy using optional stopping/Doob–Meyer. Note the time-changed filtration when checking! The last is obvious when $\langle M \rangle$ is strictly increasing, but quite tricky when $\langle M \rangle$ is constant on some intervals.]

Thm (MG as a SI): fix a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Suppose $M \in \mathcal{M}_c^2$ (so that $M_0 = 0$) and $d\langle M \rangle$ is ac wrt Lebesgue. Then there exists $X \in \mathcal{L}^*$ (i.e. $\mathbb{F}$-pm with $E[\int_0^t |X_s|^2 ds] < \infty$) and a BM W (potentially defined on a larger probability space) such that $M_t = \int_0^t X_s dW_s$ for all $t \geqslant 0$.

(Since W is defined on a larger probability space, $\mathcal{L}^* \subseteq \mathcal{L}_W^*$?)

[Work pathwise. Want to find BM W such that $dW_t = \lambda_t dM_t$. Then rearranging, we would have $dM_t = \frac{1}{\lambda_t} dW_t$, as claimed. But what is λ_t? Take bracket: $dt = \lambda_t^2 d\langle M \rangle_t$. By Radon–Nikodym, $d\langle M \rangle_t = Z_t dt$. Thus $\lambda_t = \frac{1}{\sqrt{Z_t}}$. Issue: when $Z_t = 0$! In this case construct BM B (perhaps on enlarged space) and write $dW_t = \mathbf{1}_{Z_t > 0} \lambda_t dM_t + \mathbf{1}_{Z_t = 0} dB_t$. Then $d\langle W \rangle_t = t$ (!) and in fact $M_t = \int_0^t \frac{1}{\lambda_s} dW_s$.]

0.7 Girsanov

Baby question: fix a measurable space $(\Omega, \mathcal{F})$, and fix $a \in \mathbb{R}$. If $X \sim N(0,1)$ under $\mathbb{P}$, how can I change measure to obtain $\mathbb{Q}$ such that $X \sim N(a,1)$ under $\mathbb{Q}$?

Answer: naturally the change of measure should be the ratio $\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \frac{\exp(-\frac{1}{2}(\omega - a)^2)}{\exp(-\frac{1}{2}\omega^2)}$, i.e. $\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) =$

$\exp(a\omega - \frac{1}{2}a^2)$.

Girsanov asks a harder question: given a $\mathbb{P}$-BM B, can I change measure to obtain $\mathbb{Q}$ such that B is now a $\mathbb{Q}$-BM *with drift*?

We present Girsanov in 1-d.

Thm: fix a *finite* time horizon T. Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let B be an $(\mathbb{F}, \mathbb{P})$-BM. Let $X \in \mathcal{L}^*$. Define the process Z as the stochastic exponential of $dM_t = X_t dB_t$, namely $Z_t = \exp(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t |X_s|^2 ds)$ for $t \in [0,T]$. *Assume* that Z is a genuine MG.

Define the process W by $W_t = B_t - \int_0^t X_s ds$. Define the probability measure $\mathbb{Q}$ by $d\mathbb{Q}(\omega) = Z_T(\omega) d\mathbb{P}(\omega)$. Then W is a $\mathbb{Q}$-BM.

[It is **not** in general true that Z is a MG; it is only guaranteed to be a local MG!]

[Multidimensional version: fix T, $(X^1, \ldots, X^d) \in \mathcal{L}^*$, B is a d-dimensional $\mathbb{P}$-BM, $Z_t = \exp(\sum_i \int_0^t X_s^i dB_s^i - \frac{1}{2}\sum_i \int_0^t |X_s^i|^2 ds)$. Then if Z is a genuine MG on $[0,T]$, the process $W = (W^1, \ldots, W^d)$ with components $W_t^i = B_t^i - \int_0^t X_s^i ds$ is a $\mathbb{Q}$-BM.]

Girsanov says that if we change measure from $\mathbb{P}$ to $\mathbb{Q}$, then B goes from a BM to a *drifted* BM. Simplest case: $X_s \equiv \mu$ and B is the canonical process $B(\omega) = \omega$ on the canonical space $(\mathcal{C}([0,T]), \mathcal{B}, \mathbb{P})$, where $\mathbb{P}$ is the Wiener measure. (In my mind, I imagine $\mathbb{P}$ as a "normal distribution" centered on the line $y = 0$.) Then $d\mathbb{Q} = \exp(\mu T - \frac{1}{2}\mu^2 T) d\mathbb{P}$. (In my mind, I imagine $\mathbb{Q}$ as a "normal distribution" centered on the line $y = \mu$; it places more mass on drifted sample paths.) Under $\mathbb{Q}$, $B_t = \mu t + W_t$, so that B has drift μ. (Since $B(\omega) = \omega$, $\mathbb{Q}$ coincides with the $\mathbb{Q}$-law of B.)

A typical use of Girsanov is to compute expectations like $\mathbb{E}^{\mathbb{P}}[\text{blah}] = \mathbb{E}^{\mathbb{Q}}[\frac{1}{Z_T} \cdot \text{blah}]$.

Running maximum of BM, trick from HW, add...

The most convenient condition to check that Z is a genuine MG on $[0,T]$ is the following.

Thm (Novikov condition): if $X \in \mathcal{L}^*$ such that $E[\exp(\frac{1}{2}\int_0^T |X_s|^2 ds)] < \infty$, then Z is a genuine MG on $[0,T]$.

Sketch proof of Girsanov, add...

0.8 Stochastic differential equations

Two notions of solution: strong and weak.

SDE: $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$ with initial condition $X_0 = x_0$.

Defn: given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a $\mathbb{P}$-BM W with completed natural filtration $\mathbb{F}$, a strong solution is an $\mathbb{F}$-adapted process X such that X obeys the SDE for all $t \geqslant 0$ $\mathbb{P}$-a.s., and such that we have the integrability condition $\int_0^t |b(s, X_s)| ds + \int_0^t |\sigma(s, X_s)|^2 ds < \infty$ $\mathbb{P}$-a.s. for all

$t \geqslant 0$.

A strong solution is defined in terms of the given driving BM W. In the simplest case, $X_t = f(W_t)$ for some deterministic function f.

Defn: strong uniqueness holds for the SDE if for any pair of strong solutions $X, \tilde{X}$ we have $X_t = \tilde{X}_t$ for all $t \geqslant 0$ $\mathbb{P}$-a.s.

Existence and uniqueness results for strong solutions are very similar to the theory of ODEs. Recall that a function $g(t,x)$ is locally Lipschitz in space if for all R, there exists C_R such that $|g(t,x) - g(t,y)| \leqslant C_R|x-y|$ for all $|x|, |y| \leqslant R$ and for all t.

Thm: if b, σ are locally Lipschitz in space, then we have strong uniqueness of the SDE.

[Imitate uniqueness for ODEs. Localize such that $|X_t|, |\tilde{X}_t| \leqslant R$ for $t \leqslant \tau_R$, take the difference, compute $E[|X_t - \tilde{X}_t|^2]$ via Ito, and apply Gronwall.]

Thm: assume that b, σ are (globally) Lipschitz in space, and assume that we have the growth condition $|b(t,x)|^2 + |\sigma(t,x)|^2 \leqslant C(1+|x|^2)$. Then we have strong existence for the SDE.

The proof illustrates the power of the BDG inequality, our go-to maximal inequality.

[Picard iterations. Begin with $X^{(0)} \equiv x_0$, then iterate via the recurrence $X_t^{(n+1)} = x_0 + \int_0^t b(s, X_s^{(n)})ds + \int_0^t \sigma(s, X_s^{(n)})dW_s$ for all $n \geqslant 0$. One can show via BDG that $X^{(n)}\big|_{[0,T]}$ is a Cauchy sequence with respect to the uniform norm, hence it converges to a continuous process $X^{(\infty)}$, and $X^{(\infty)}$ is a fixed point to the iteration.]

Now a relaxed assumption for uniqueness, when σ is not Lipschitz.

Thm (Yamada-Watanabe I): assume b is Lipschitz and that $|\sigma(t,x) - \sigma(t,y)| \leqslant h(|x-y|)$, where h is a strictly increasing function with $h(0) = 0$ and $\int_0^\varepsilon h(u)^{-2}du = \infty$ for all $\varepsilon > 0$. Then we have strong uniqueness for the SDE.

Note that $h(u) = \sqrt{u}$ is the largest power law that obeys this condition: $\int_0^\varepsilon u^{-1}du = \infty$.

[Do not consider $E[|X_t - \tilde{X}_t|^2]$, but instead consider $E[|X_t - \tilde{X}_t|]$. Apply generalized Ito rule for convex functions.]

The following is a qualitative comparison principle. It states that if the strong solutions $X, \tilde{X}$ share *the same volatility* and if the drift of X is less than the drift of $\tilde{X}$, then under the assumption of strong uniqueness, if X begins below $\tilde{X}$, it must always remain below $\tilde{X}$. (Think back to the power of comparison principles in geometry.)

Thm (comparison principle): suppose $X, \tilde{X}$ are strong solutions of the SDEs $dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t$, $d\tilde{X}_t = \tilde{b}(t,\tilde{X}_t)dt + \sigma(t,\tilde{X}_t)dW_t$ with initial conditions $X_0 = x_0$ and $\tilde{X}_0 = \tilde{x}_0$. Assume

- (b,σ) and $(\tilde{b},\sigma)$ are continuous and at least one pair satisfies the assumption of Yamada-Watanabe I;
- $x_0 \leqslant \tilde{x}_0$;

- $b(t,x) \leqslant \tilde{b}(t,x)$ for all $t \geqslant 0$ and $x \in \mathbb{R}$.

Then $X_t \leqslant \tilde{X}_t$ for all $t \geqslant 0$ $\mathbb{P}$-a.s.

[Assume strong uniqueness holds for $(\tilde{b}, \sigma)$. Assume $b = \tilde{b}$. (Indeed, if $b < \tilde{b}$, then replacing b with $\tilde{b}$ yields a process $\hat{X}$ that dominates X.) If with positive probability X and $\tilde{X}$ cross at some time t, we could concatenate $X|_{[0,t]}$ and $\tilde{X}|_{[t,\infty)}$ to obtain a new strong solution to the SDE $(\tilde{b}, \sigma)$, contradiction to strong uniqueness. Alternate proof: apply Ito and Gronwall to $E[(\tilde{X}_t - X_t)_+]$.

We now consider weak solutions, weak uniqueness, and weak existence.

Defn: a weak solution to the SDE is a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ with an $(\tilde{\mathbb{F}}, \tilde{\mathbb{P}})$-BM $\tilde{W}$ and an $\tilde{\mathbb{F}}$-pm process X such that X obeys the SDE for all $t \geqslant 0$ $\mathbb{P}$-a.s., and such that we have the integrability condition $\int_0^t |\tilde{b}(s, X_s)| ds + \int_0^t |\tilde{\sigma}(s, X_s)|^2 ds < \infty$ $\tilde{\mathbb{P}}$-a.s. for all $t \geqslant 0$.

In the notion of weak solution, we get to choose the probability space and the BM. We need not work with the probability space and the BM provided to us. (Think back to the squared Bessel process.)

Defn: we have pathwise uniqueness if for any pair of weak solutions $X, \tilde{X}$ *on the same probability space*, $X = \tilde{X}$ $\mathbb{P}$-a.s.

Pathwise uniqueness is restrictive. We prefer the following notion.

Defn: we have weak uniqueness if for any pair of weak solutions $X, \tilde{X}$ (possibly on different probability spaces), $X \stackrel{d}{=} \tilde{X}$.

Thm: consider the simplified SDE $dX_t = b(t, X_t)dt + \sigma dW_t$ on the finite time horizon $t \in [0, T]$ with initial condition $X_0 = x$. (Here σ is a *constant*.) If b is bounded measurable, then we have weak existence and uniqueness.

[Idea: use Girsanov to change measure and eliminate drift b. Since b bounded, Novikov holds, and Z_T is genuine MG. For uniqueness, of course check $X \stackrel{d}{=} \tilde{X}$ wrt to the *original* measures $\mathbb{P}, \tilde{\mathbb{P}}$; i.e. for every Borel $A \subseteq \mathcal{C}([0,T])$, show $\mathbb{P}(X|_{[0,T]} \in A) = \tilde{\mathbb{P}}(\tilde{X}|_{[0,T]} \in A)$ by changing measure.]

The following is a powerful result. If all weak solutions happened to live on the same probability space, then strong uniqueness would certainly imply weak uniqueness (equal a.s. implies equal in distribution). But weak solutions are allowed to live on different spaces, so the claim is surprising.

Thm (Yamada-Watanabe II): strong uniqueness implies weak uniqueness.

[Recall the Skorokhod representation from probability theory: given a pair of real-valued random variables $X, \tilde{X}$ that are equal in distribution but which live on possibly different probability spaces, one can couple $X, \tilde{X}$ by constructing $Y, \tilde{Y}$ on the *same* probability space such that $Y \stackrel{d}{=} X$, $\tilde{Y} \stackrel{d}{=} \tilde{X}$, and (most importantly) $Y = \tilde{Y}$ a.s. One can do the same for processes $X, \tilde{X}$: put them both on $(\mathcal{C}([0,\infty)), \mathcal{B}, \mathbb{P}^*)$ for some $\mathbb{P}^*$. If $Y, \tilde{Y}$ satisfy the original SDE, we deduce $Y \stackrel{d}{=} \tilde{Y}$, hence $X \stackrel{d}{=} \tilde{X}$.]

There is one last approach to constructing weak solutions: the martingale problem. In some sense, it is a generalization of Levy's characterization of BM. Levy states that if $M \in \mathcal{M}_c^2$ and $\langle M \rangle_t = t$, then M has the distribution of BM. The martingale problem allows us to characterize the law of a weak solution of an SDE that is more general than $dX_t = dW_t$.

We work in $\mathbb{R}$, but this can be generalized to $\mathbb{R}^d$, where the drift b is a d-dimensional vector and the volatility σ is a $d \times d$ matrix.

Consider a weak solution to the SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$. Given $f \in \mathcal{C}^2(\mathbb{R})$, define the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{A}f)(s, X_s)ds \tag{13}$$

where the generator is given by $\mathcal{A} = b\partial_x + \frac{1}{2}\sigma^2\partial_{xx}$ in 1-dimension. (In $\mathbb{R}^d$, $\mathcal{A} = \sum_i b_i \partial_{x_i} + \frac{1}{2}\sum_{i,j,k} \sigma_{ik}\sigma_{jk}\partial_{x_i x_j}$.)

The operator $\mathcal{A}$ is specially chosen to kill all drift in M^f, so that M^f is a local martingale. Indeed, by Ito

$$\begin{aligned} dM_t^f &= df(X_t) - (\mathcal{A}f)(t, X_t) && (14)\\ &= \partial_x f(X_t)dX_t + \frac{1}{2}\partial_{xx} f(X_t)d\langle X \rangle_t && (15)\\ &= \partial_x f(X_t)\sigma(t, X_t)dW_t && (16) \end{aligned}$$

Defn: if $\mathbb{P}$ is a probability measure on the canonical space such that M^f is a continuous (local) MG for all $f \in \mathcal{C}^2(\mathbb{R})$, then $\mathbb{P}$ is a solution to the (local) MG problem.

[Clarification: think of X as merely a measurable map; for instance, the canonical process $X(\omega) = \omega$ on $\mathcal{C}([0, \infty))$. We must provide the probability measure $\mathbb{P}$.]

Prop (shown above): if $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, X)$ is a weak solution to the SDE, then the pushforward of $\mathbb{P}$ by X is a solution to the local MG problem.

Prop (converse): let X be a continuous process. If $\mathbb{P}$ is a measure on the canonical space that solves the local MG problem, then X is a weak solution to the SDE.

[Goal: find a probability space carrying a BM W such that X solves the given SDE. Plug $f(x) = x$ and $g(x) = x^2$ into the local MG problem. In differential form, we obtain $dX_t = dM_t + bdt$ and $dX_t^2 = dN_t + (bX_t + \sigma^2)dt$. The first equation is promising: if only $dM_t = \sigma dW_t$ for some BM W! By Ito, the second equation becomes $dN_t = 2X_t dM_t + d\langle M \rangle_t - \sigma^2 dt$. Rearranging, $dN_t - 2X_t dM_t = d\langle M \rangle_t - \sigma^2 dt$. Since the LHS is local MG and the RHS is BV, both sides are zero. Thus $d\langle M \rangle_t = \sigma^2 dt$! M has the right QV, but we don't yet have a BM. Apply MG representation theorem to obtain $dM_t = Z_t d\tilde{W}_t$, where $\tilde{W}$ is a $\tilde{\mathbb{P}}$-BM on some enlarged probability space. Taking QV, we find $Z = \sigma$, so that $dX_t = \sigma d\tilde{W}_t + bdt$, as desired.]

So far we only have a weak existence result for σ constant. The MG problem yields the following.

Thm (Stroock-Varadhan): if b, σ are continuous and bounded, then we have weak existence for the SDE.

Example of using MG problem to check weak solution?, add...

0.9 Link to PDE

Weak and strong solutions to SDEs relate to solutions to certain parabolic PDEs.

First, weak solutions.

Thm: consider the *time-homogeneous* SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Consider the Cauchy problem $(\partial_t - \mathcal{A})u = 0$ with *initial* condition $u(0,x) = g(x)$. Then

$$\text{classical soln to PDE } \forall g \in \mathcal{C}_c^\infty(\mathbb{R}) \implies \text{weak uniqueness for SDE} \tag{17}$$

[Consider two weak solutions $X, \tilde{X}$. We just check $X_t \stackrel{d}{=} \tilde{X}_t$; finite-dimensional marginals are very hard. Fix t, and reverse time to obtain $(\partial_s + \mathcal{A})w = 0$ on $[0,t]$ with terminal condition $w(t,x) = g(x)$. Trick: apply multidimensional Ito to the semiMG $w(s, X_s)$ and collect terms to obtain

$$w(t, X_t) = w(0, X_0) + \int_0^t \partial_x w(s, X_s)\sigma(X_s)dW_s \tag{18}$$

where we used $\partial_s + \mathcal{A} = 0$ to kill the drift terms. If we localize X, then the SI becomes a genuine MG. Taking the expectation, the SI dies and we obtain $E[g(X_t)] = w(0, X_0)$. The same holds for $\tilde{X}$. Since $X, \tilde{X}$ share the same IC, $E[g(X_t)] = E[g(\tilde{X}_t)]$ for all $g \in \mathcal{C}_c^\infty$, so $X_t \stackrel{d}{=} \tilde{X}_t$.]

Next, strong solutions. This is the famous Feynman-Kac formula.

Let T be a finite time horizon. Fix $(t,x) \in [0,T] \times \mathbb{R}$. Let $X^{t,x}$ be a *strong* solution to the SDE $dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s$ for $s \in [t,T]$ started at $X_t = x$. Define the *value function* by the conditional expectation

$$v(t,x) = E[e^{-\int_t^T k(u,X_u)du} g(X_T) + \int_t^T e^{-\int_t^s k(u,X_u)du} f(s, X_s)ds | X_t = x] \tag{19}$$

where k is the instantaneous discount factor, f is the running reward, and g is the terminal reward. (We sometimes abbreviate $K_{t,x}(s) = e^{-\int_t^s k(u,X_u)du}$.) (An "average over all paths" to time T, starting from (t,x).)

Consider the seemingly unrelated parabolic PDE $ku - \mathcal{L}u - f = 0$ for $(t,x) \in [0,T] \times \mathbb{R}$ with *terminal* condition $u(T,x) = g(x)$. Here $\mathcal{L} = \partial_t + \mathcal{A}$ is a time-dependent generator (Note the annoying sign conventions.)

Assumption: f, g, k, b, σ continuous, k bounded below by a finite constant, and f, g of polynomial growth (!). (Less restrictive conditions in K-S, add...)

Prop: if $u \in \mathcal{C}^0([0,T] \times \mathbb{R}) \cap \mathcal{C}^{1,2}([0,T) \times \mathbb{R})$ solves the PDE, then $u = v$.

Prop: if the value function v is $\mathcal{C}^{1,2}([0,T] \times \mathbb{R})$, then v solves the PDE.

$$\text{smooth value function} \iff \text{smooth soln to PDE} \tag{20}$$

How to check the conditions of the Props? (!) add...

[($\leftarrow$) Suppose u solves PDE. Want to represent it as conditional expectation. Trick: apply Ito product rule and multidimensional Ito to $K_{t,x}(s) \cdot u(s, X_s)$. The drift terms $-kv + \mathcal{L}v$ will simplify to $-f$ from the PDE, so $d(K_{t,x}u) = K_{t,x}(-fds + \sigma\partial_x u dW_s)$. Writing in integral form, *localizing*, and taking expectation, the SI dies and we obtain

$$E[K_{t,x}(\tau^n)u(\tau^n, X_{\tau^n})|X_t = x] - u(t,x) = -E[\int_t^{\tau^n} K_{t,x}(s)f(s, X_s)ds|X_t = x] \tag{21}$$

so sending $n \to \infty$, $u(t,x) = E[K_{t,x}(T)g(X_T) + \int_t^T K_{t,x}(s)f(s, X_s)ds|X_t = x]$ as claimed.

($\rightarrow$) Suppose the value function is $\mathcal{C}^{1,2}$ (of course, for some strong solution $X^{t,x}$). We use the dynamic programming principle: if τ is a stopping time taking values in $[t, T]$, then the value function obeys

$$v(t,x) = E[K_{t,x}(\tau)v(\tau, X_\tau) + \int_t^\tau K_{t,x}(s)f(s, X_s)ds|X_t = x] \tag{22}$$

I.e. you can use the terminal condition $v(\tau, X_\tau)$ at time τ, instead of at time T. DPP is just tower property and time consistency of K. Apply Ito to $K_{t,x}(s) \cdot v(s, X_s)$ as before to obtain

$$v(t,x) = E[K_{t,x}(\tau)v(\tau, X_\tau) - \int_t^\tau K_{t,x}(s)(-kv + \mathcal{L}v)ds|X_t = x] \tag{23}$$

Compare with DPP, take $\tau^n \searrow 0$, and apply Lebesgue differentiation to obtain $kv - \mathcal{L}v = f$.]

Kakutani's solution to the elliptic time-independent Dirichlet problem, add...

see practice midterm+final/finish hw/examples...

Be able to derive Feynman-Kac, add...

0.10 Examples

Examples often arise as "solutions" to SDEs. SDEs are often stochastic variants of familiar ODEs from mechanics.

Silly example: $f(x) = x^2$, Itô process $dX_t = b_t dt + \sigma_t dW_t$. Then if $Y = f(X)$, $dY_t = 2X_t dX_t + 2d\langle X\rangle_t = 2X_t(b_t dt + \sigma_t dW_t) + 2\sigma_t^2 dt = (2b_t X_t + 2\sigma_t^2)dt + 2X_t\sigma_t dW_t$.

Itô product rule: if X, Y are continuous semiMGs, then $d(XY)_t = X_t dY_t + Y_t dX_t + d\langle X, Y\rangle_t$. (Apply Itô to $f(x,y) = xy$ and $Z = (X, Y)$. Then $dZ_t = X_t dY_t + Y_t dX_t + 2 \cdot \frac{1}{2} d\langle X, Y\rangle_t$; note the $\cdot 2$!)

Function of time and space: let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a function of (time, space). (For instance, $e^{-t}X_t$ or e^{tX_t}.) Then if $Y_t = t$, we can apply Itô to f and $Z = (Y, X)$. Since Y_t is BV,

$$dZ_t = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}\partial_{xx} f(t, X_t)d\langle X\rangle_t \tag{24}$$

Integrating (don't forget IC!),

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_t f(s, X_s)ds + \int_0^t \partial_x f(s, X_s)dX_s + \frac{1}{2}\int_0^t \partial_{xx} f(s, X_s)d\langle X\rangle_s \tag{25}$$

Exponential local MG: given $M \in \mathcal{M}_c^2$, $X \in \mathcal{L}_M^*$, define

$$Y_t = \exp(\int_0^t X_s dM_s - \frac{1}{2}\int_0^t X_s^2 d\langle M\rangle_s) \tag{26}$$

Claim $dY_t = X_t Y_t dM_t$ with IC $Y_0 = 1$. Thus $Y_t = 1 + \int_0^t X_s Y_s dM_s$ is certainly a loc MG, but not necessarily a MG! To show SDE, apply Itô to $f(x) = \exp(x)$ and the process $dZ_t = X_t dM_t - \frac{1}{2}X_t^2 d\langle M\rangle_t$. Then $dY_t = Y_t dZ_t + \frac{1}{2}Y_t d\langle Z\rangle_t = Y_t dZ_t + \frac{1}{2}Y_t X_t^2 d\langle M\rangle_t = Y_t X_t dM_t$, as claimed.

Doleans–Dade exponential ("stochastic exponential"): a generalization of an exponential loc MG. Given a continuous semiMG Z, the solution to the SDE $dY_t = Y_t dZ_t$ with IC $Y_0 = 1$ is given as $Y_t = \exp(Z_t - Z_0 - \frac{1}{2}\langle Z\rangle_t)$. (Sometimes denoted $\mathcal{E}(Z)$.) Indeed, apply Itô to $f(x) = \exp(x)$ and the process $dX_t = dZ_t - \frac{1}{2}d\langle Z\rangle_t$ to find $dY_t = Y_t dX_t + \frac{1}{2}Y_t d\langle X\rangle_t = Y_t dZ_t - \frac{1}{2}Y_t d\langle Z\rangle_t + \frac{1}{2}Y_t d\langle Z\rangle_t = Y_t dZ_t$, as claimed. In the last step we used the fact that $\langle Z\rangle$ is nondecreasing, hence BV.

Geometric BM: $S_t = S_0 \exp(\sigma W_t + (\mu - \frac{\sigma^2}{2})t)$, where W is a standard BM. Note that geometric BM is the Doleans–Dade exponential of the Itô process $dZ_t = \mu dt + \sigma dW_t$ with IC $Z_0 = -\log S_0$.

Bessel squared process: $dR_t = m dt + 2\sqrt{R_t} dW_t$, where $m \in \mathbb{N}$. A "weak" solution to this equation is given as $R_t = \sum_{i=1}^m |B_t^i|^2$, where $B = (B^1, \ldots, B^m)$ is standard m-dim BM. Apply Itô to $f(x) = x_1^2 + \ldots + x_m^2$ and B to find $dR_t = \sum_i 2B_t^i dB_t^i + \frac{1}{2}\sum_i 2dt = m dt + 2\sqrt{R_t} \cdot \sum_i \frac{1}{\sqrt{R_t}} B_t^i dB_t^i$. We check via Lévy that $Z_t := \int_0^t \sum_i \frac{1}{\sqrt{R_s}} B_s^i dB_s^i$ is a BM. First note Z is a genuine square integrable MG, since B_s^i is in $\mathcal{M}_c^2$ and the integrand $\frac{B_s^i}{\sqrt{R_s}}$ is bounded. By independence of B^i's, $d\langle Z\rangle_t = \sum_i \langle \frac{1}{\sqrt{R}} B^i dB^i\rangle_t = \sum_i \frac{(B_t^i)^2}{R_t} dt = dt$. (In integral form, $\langle Z\rangle_t = \sum_i \langle \int_0^\cdot \frac{1}{\sqrt{R_s}} B_s^i dB_s^i\rangle_t = \sum_i \int_0^t (\frac{1}{\sqrt{R_s}} B_s^i)^2 dt = t$.) Thus Z is a BM.

Inverse Bessel process: $Y = (|x + W_t|^{-1})_{t \geqslant 0}$, where $x \in \mathbb{R}^3 \setminus 0$ and W is 3-d BM. Note that $y \mapsto \frac{1}{|y|}$ is a harmonic function! Since the components of W are independent, Itô's formula reduces to

$$dY_t = \nabla f(X_t) \cdot dX_t + \frac{1}{2}\triangle f(X_t) dt = \nabla f(X_t) \cdot dW_t \tag{27}$$

Since the LS integral vanishes, since W is a MG, and since $\nabla f(X_t)$ has the requisite integrability, we deduce that Y is a *loc MG*! More explicitly, we may write $Y_t = \frac{1}{|x|} + \int_0^t \nabla f(X_s) \cdot dW_s$ and recall that a SI wrt a MG is a loc MG. Indeed, this was one of the first examples of a loc MG that is *not* a genuine MG (because $\lim_{t\to\infty} Y_t = 0$ by the transience of BM in $\mathbb{R}^3$).

OU process: a SP with a "restoring force" and noise. (If we imagine X_t is velocity at time t, then the restoring force is a drag: $F = -av$.) Consider the SDE $dX_t = -aX_t dt + \sigma dW_t$, where $a, \sigma > 0$ and W is a standard BM. We claim that $X_t = xe^{-at} + \sigma \int_0^t e^{-a(t-s)} dW_s$ is a solution; note the additive separability of the deterministic and stochastic parts of the solution. (The SI $\int_0^t e^{-a(t-s)} dW_s$ is also interesting in that it is a rolling average of previous noise increments, with highest weight placed on the most recent noise.) In this particular case, it is more convenient to check the *integral* form of the SDE, namely $X_t = X_0 + \int_0^t -aX_s ds + \int_0^t \sigma dW_s$.

Tanaka SDE, add...

How to "guess" a SI: find a function $f(t, x)$ such that $\partial_x f(t, X_t) dX_t$ is exactly the integrand. Then applying Itô to $f(t, X_t)$ and rearranging will do the trick.

0.11 Problem solving tips

- check integrability conditions for strong/weak solutions? ask...

- be careful of the annoying signs when using Feynman-Kac...

- check existence and uniqueness conditions for Feynman-Kac?...

—

- Q: assume $\mathcal{M}_c^2$ means $M_0 = 0$?, add...

- Q: the LS integral is a BV process? conditions? add...

- Q: we allowed to write cross-variation in differential form, but be careful with notation

- Q: when checking an SDE for a solution that involves an integral representation, it might be easier to work in integral form (see OU)

- when checking MG, check (1) integrable, (2) adapted, (3) MG property

- a MG has uncorrelated increments

- if computing bracket, check SP is in $\mathcal{M}_c^2$. (continuity will follow from SI being continuous)

- trick: if $M \in \mathcal{M}_c^2$, then $E[(M_t - M_s)^2|\mathcal{F}_r] = E[M_t^2 - M_s^2|\mathcal{F}_r]$

- to show $M \in \mathcal{M}_c^2$ is constant, show that bracket is identically zero

- recall Doob–Meyer: if $\langle M\rangle$ is continuous nondecreasing and $M^2 - \langle M\rangle$ is MG, then $\langle M\rangle$ is the QV. (Note that it's M^2, *not* M!)

- Itô isometry: "you can move the bracket inside, but square the integrand and put a bracket on the differential!"

- don't forget that bracket of a sum must be treated via *cross-variation!*

- cross-variation of independent processes is zero; note this *before* starting Itô computations!

- warning: apriori, a SI of the form $\int X dM$ where $M \in \mathcal{M}_c^2$ and $\int_0^\infty X^2 d\langle M\rangle < \infty$ a.s. is merely a loc MG; only when $X \in \mathcal{L}_M^*$ can we deduce that $\int X dM$ is MG. Ex: if X is bounded, then $X \in \mathcal{L}_M^*$

- when checking $\mathcal{L}_M^*$, must check integrability assumption *and* $\mathbb{F}$-progressively measurable!

- if a SP (after applying Itô's formula) has all LS terms die, then it is a loc MG. (But must make sure it is fully expanded, i.e. we are really left with a SI wrt a loc MG! The innocuous dZ_t might be hiding a semiMG with drift!)

- don't forget $\frac{1}{2}$ in second-order Itô term, and be careful with repetition of i, j in multi-d Itô's formula. In particular case of Itô for $f(t, x)$, check $f \in C^{1,2}$ assumption

- if you need to obtain a BM, consider using DDS thm, or by checking $\mathcal{M}_c^2$ + Lévy characterization

- if you have computed $d\langle M\rangle$ and find that it is ac with $d\langle M\rangle = Z dt$, then you can write $M = \int X dW$, where $X = \sqrt{Z}$