ORF526 Final Cheatsheet

Sunay Joshi

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Markov's Inequality: $\mathbb{P}(X \geqslant x) \leqslant \frac{\mathbb{E}[X]}{x}$.

Chebyshev's Inequality: $\mathbb{P}(|X - \mathbb{E}[X]| \geqslant \lambda\sigma) \leqslant \frac{1}{\lambda^2}$.

Jensen's Inequality: if g is convex, then $\mathbb{E}[g(X)] \geqslant g(\mathbb{E}[X])$.

Holder's Inequality: if $\frac{1}{p} + \frac{1}{q} = 1$, then $\mathbb{E}[|XY|] \leqslant \mathbb{E}[|X|^p]^{1/p} \cdot \mathbb{E}[|Y|^q]^{1/q}$.

MON: if $0 \leqslant X_n \nearrow X_\infty$ a.s., then $\mathbb{E}[X_n] \to \mathbb{E}[X_\infty]$.

DCT: if $X_n \xrightarrow{as} X_\infty$ for X_n mb and $|X_n| \leqslant Y$ a.s. with $\mathbb{E}[Y] < \infty$, X_n is mb and $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

Fatou's Lemma: if $X_n \geqslant Y$ with $\mathbb{E}[|Y|] < \infty$, then $\mathbb{E}[\liminf X_n] \leqslant \liminf \mathbb{E}[X_n]$. *(+ for events!)*

Reverse Fatou's lemma: if $X_n \leqslant Y$ with $\mathbb{E}[|Y|] < \infty$, then $\mathbb{E}[\limsup X_n] \geqslant \limsup \mathbb{E}[X_n]$.

Note that DCT provides a situation where a.s. implies L^1. Another way:

Scheffe's Lemma: if $X_n \xrightarrow{as} X_\infty$ and $\mathbb{E}[|X_n|] \to \mathbb{E}[|X_\infty|]$, then $X_n \xrightarrow{L^1} X_\infty$.

How probability can imply L^1:

Defn: $\{X_\alpha\}$ is uniformly integrable (u.i.) if $\lim_{M\to\infty} \sup_\alpha \mathbb{E}[|X_\alpha| \mathbf{1}_{|X_\alpha| \geqslant M}] = 0$.

Ex: for a u.i. sequence $X_n = \frac{n}{\log n} \mathbf{1}_{(0, \frac{1}{n})}$ on $\Omega = (0, 1]$.

Vitali: if $X_n \xrightarrow{p} X_\infty$, then $\{X_n\}$ is u.i. iff $X_n \xrightarrow{L^1} X_\infty$.

Change of variables: $\mathbb{E}[h(X)] = \int_{\mathbb{R}} h(x) d\mathcal{P}_X(x)$ for any h. (Very useful! Don't recompute CDF!)

Integration by Parts: if $X \geqslant 0$, then $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > x) dx$.

Kolmogorov 0-1 Law: if $\{X_n\}$ m.i., then tail σ-algebra τ is $\mathbb{P}$-trivial.

Weak LLN (L^2): X_n pairwise uncorrelated, $\mathrm{Var}(X_i) \leqslant C$, $\mathbb{E}X_i = x$. Then $\frac{1}{n} \sum X_i \xrightarrow{L^2} x$.

Weak LLN (p): X_n iid s.t. $x\mathbb{P}(|X_1| > x) \to 0$. Then $\frac{1}{n}\sum X_i - \mathbb{E}[|X_1| \cdot \mathbf{1}_{|X_1| \leqslant n}] \xrightarrow{p} 0$.

- (B-C I): $\sum \mathbb{P}(A_i) < \infty \implies \mathbb{P}(A_n \text{ i.o.}) = 0$
- (B-C II): **if m.i.,** then $\sum \mathbb{P}(A_n) = \infty \implies \mathbb{P}(A_n \text{ i.o.}) = 1$

Strong LLN: X_n pairwise independent, identically distributed, $X_n \in L^1$. Then $\frac{1}{n}\sum X_i \xrightarrow{a.s.} \mathbb{E}X_1$.

CLT (Laplace): $\{X_n\}$ iid with $\mathbb{E}X_1 = \mu$, $\text{Var}X_1 = v$. Then $\mathbb{P}(\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{nv}} \leqslant a) \to \Phi(a)$.

Skorokhod Representation Thm: if $X_n \xrightarrow{\mathcal{D}} X$, then there exists $(\Omega, \mathcal{F}, \mathbb{P})$, Y_n, Y s.t. $Y_n \sim X_n$, $Y \sim X$; and $Y_n \xrightarrow{a.s.} Y$ *(an upgrade!).*

Weak convergence: $X_n \xRightarrow{w} X_\infty$ if $\mathbb{E}f(X_n) \to \mathbb{E}f(X_\infty)$ for all $f \in C_b^0$.

Portmanteau Thm: $\nu_n \xRightarrow{w} \nu_\infty$ iff $\forall$ closed F, $\limsup \nu_n(F) \leqslant \nu_\infty(F)$.

Defn: $\{X_\alpha\}$ tight if $\forall \varepsilon$, $\exists M_\varepsilon$ s.t. $\mathbb{P}(X_\alpha \notin [-M_\varepsilon, M_\varepsilon]) \leqslant \varepsilon \quad \forall \alpha$.

Defn: characteristic function $\Phi_X(\theta) = \mathbb{E}[e^{i\theta X}]$. *(Normal: $e^{-\theta^2/2}$)*

Props: $\Phi_{aX+b}(\theta) = e^{ib\theta}\Phi_X(a\theta)$; $X \perp Y$ implies $\Phi_{X+Y} = \Phi_X\Phi_Y$; $\Phi_X = \Phi_Y$ implies $\mathcal{P}_X = \mathcal{P}_Y$.

Levy continuity: weak convergence $\implies$ convergence of Φ_{ν_n}; convergence of Φ_{ν_n} + continuity of Φ_{ν_∞} at 0 $\implies$ weak convergence.

Defn: if $X \in L^1$, $Y = \mathbb{E}[X|G]$ is conditional expectation if Y is G-mb and $\mathbb{E}[Y \cdot \mathbf{1}_A] = \mathbb{E}[X \cdot \mathbf{1}_A] \quad \forall A \in G$. *("best guess given G, a rv")*

Props: (Tower) $\mathbb{E}[\mathbb{E}[X|G]|H] = \mathbb{E}[X|H]$ if $H \subseteq G$, ($\perp$) $\mathbb{E}[X|G] = \mathbb{E}X$, (mb) $\mathbb{E}[X|G] = X$.

Defn: $(X_n), (\mathcal{F}_n)$ martingale if $X_n \in L^1$, (X_n) adapted, $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = X_{n-1}$. *(check all 3!)*

Sub-martingale = increasing, super-martingale = decreasing.

Props: (X_n) MG + Φ convex implies $(\Phi(X_n))$ sub-MG; (X_n) sub-MG + Φ convex nondecreasing implies $(\Phi(X_n))$ sub-MG.

Defn: τ stopping time if $\{\tau \leqslant n\} \in \mathcal{F}_n \quad \forall n$.

Props: (X_n) (sub-)MG implies $(X_{n\wedge\tau})$ (sub-)MG; $\theta \leqslant \tau$ implies $\mathbb{E}X_{n\wedge\theta} \leqslant \mathbb{E}X_{n\wedge\tau}$.

Decomposition: $X_n = A_n + Y_n$ where A_n predictable, Y_n MG.

Maximal inequality: (X_n) sub-MG implies $\mathbb{P}(\max_{k\in[n]} X_k > x) \leqslant \frac{1}{x}\mathbb{E}[(X_n)_+]$. *(Like Markov!)*

MG Convergence: if (X_n) sup-MG, $\sup \mathbb{E}[(X_n)_-] < \infty$, then $X_n \xrightarrow{as} X_\infty$ and $\mathbb{E}|X_\infty| \leqslant \liminf \mathbb{E}|X_n|$.

Thm: $\{X_n\}$ sub-MG. Then u.i. iff $X_n \xrightarrow{L^1} X_\infty$.

Optional stopping: (Y_n) sub-MG, $\theta \leqslant \tau$ stopping times. Then $\mathbb{E}Y_\tau \geqslant \mathbb{E}Y_\theta \geqslant \mathbb{E}Y_0$. *(= if MG!)*

Distributions: $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$, $\lambda e^{-\lambda x}$, $e^{-\mu} \frac{\mu^k}{k!}$.