Math Colloquium: Professor Deligne

Math Club

July 3, 2026

Topic: Algebraic varieties

For our purposes, an *algebraic variety* is the solution set to a polynomial equation. We work over $\mathbb{C}$, because the theory over $\mathbb{R}$ is more technical.

Examples:

- In $\mathbb{C}^2$, $y^2 = x^3 + x$ is a torus minus a point (at infinity).
- In $\mathbb{P}^2(\mathbb{C})$, the homogenized equation $y^2z = x^3 + xz^2$ is a torus. It is isomorphic to $\mathbb{C}/\mathbb{Z}[i]$. Here i corresponds to the automorphism given by multiplication by $(-1, i, 1)$ (?).
- In $\mathbb{C}^3$, $x^2 + y^3 + z^5 = 0$ is a cone. To see this, note that given $(x_0, y_0, z_0) \in S^3$, the tuple $(t^{15}x_0, t^{10}y_0, t^6z_0)$ lies on $x^2 + y^3 + z^5 = t^{30}$ (?).
- $w^2 + v^2 + x^2 + y^3 + z^5 = 0$ corresponds to an exotic sphere (?).

We consider projective varieties (?).

Consider a compact variety X. We may decompose it into simplices and compute its Euler–Poincare invariant as follows: $\chi(X) = \sum_{i \geqslant 0}(-1)^i \cdot (\#$ dim i simplices). For example, the circle has Euler characteristic $2 - 2 = 0$ and the sphere has Euler characteristic $3 - 3 + 2 = 2$.

We may also consider the homotopy type of X. Two spaces X, Y are homotopy equivalent if there exist homotopies $X \to Y$ and $Y \to X$ such that either composition can be deformed to the identity. Given a fibration $X \to Y$ with fiber F, the Euler characteristic is multiplicative: $\chi(X) = \chi(Y)\chi(F)$ (?).

Local properties of variety: Let $0 \in X \subseteq \mathbb{C}^n$ be a variety with a singularity (?) at 0. The *link* of X at 0 is the intersection of a small ball around 0 with X (?).

Theorem: $\chi(\text{link}) = 0$.

Proof: a rigorous proof requires "resolution of singularities". For an intuitive proof, suppose X is given as the zero set of a homogeneous polynomial $P = 0$. Note that the link L is stable under

scaling all of its coordinates. We see that there is a free action of $U^1 := \{z \in \mathbb{C} || z| = 1\} \simeq S^1$ on the link, since multiplication by $z \in U^1$ preserves the distance to the origin. We obtain a fibration $L \to L/S^1$ with fiber S^1. Therefore $\chi(L) = \chi(S^1)\chi(L/S^1) = 0 \cdot \chi(L/S^1) = 0$. ■

There is a pale reflection of this fact over $\mathbb{R}$: the Euler characteristic of any link is even. This can be seen by considering the real variety as the subset of a complex variety that is fixed by conjugation $z \mapsto \overline{z}$.

We now study two "hidden symmetries" of varieties.

Cohomology of variety: Consider X nonsingular. We can treat X as a complex manifold. Then for all $x \in X$, the tangent space T_x is a $\mathbb{C}$-vector space. We have a natural $\mathbb{C}^\times$-action on the tangent bundle TX by simply multiplying by $z \in \mathbb{C}$. In fact, we have an action on $\Omega^p = \bigwedge_{i=1}^p (T_p)^\vee_{real} = \{\text{differential p-forms on X}\}$ (which action?).

(How to write down an element of Ω^p; $\omega = \sum dz_i \wedge d\overline{z}_j$?)

Recall the exterior derivative $d : \Omega^p \to \Omega^{p+1}$. The p-th cohomology group is defined as $H^p = \ker(d : \Omega^p \to \Omega^{p+1})/\text{im}(d)$, i.e. the closed forms modulo the exact forms. The cohomology groups reveal topological properties of the space. For instance, for the punctured plane, gradθ (where θ is the polar angle) is a closed but not exact vector field, so the cohomology detects the nontrivial fundamental group.

Let us write the $\mathbb{C}^\times$ as $z * \alpha$, where $z \in \mathbb{C}^\times$ and $\alpha \in \Omega^p$. Consider the following "twisted" version of the cohomology group: $\{\alpha \in H^p | d(z * \alpha) = 0\}/\{d\beta | \beta \in H^{p-1}\}$. The key fact is that this is isomorphic to H^p (?). This $\mathbb{C}^\times$ action is the first hidden symmetry of the variety.

Action of Galois group on variety: Next, suppose X is a variety given by the zero set $P = 0$ of some homogeneous polynomial. Suppose that X has coefficients in $\mathbb{Q}$. (In general, we would consider coefficients in some number field K.) It turns out that we do not lose a lot by making this assumption; one can show that we can carefully perturb the coefficients of X to lie in a number field without affecting the topology of the variety.

Consider the set of points of X whose coefficients lie in $\overline{\mathbb{Q}}$, which we denote $X_{\overline{\mathbb{Q}}}$. There is a natural action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X_{\overline{\mathbb{Q}}}$, since applying $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $P = 0$ preserves the rational coefficients. (A modern perspective views X as a scheme over $\mathbb{Q}$ and notes that $X_{\overline{\mathbb{Q}}} = X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. (?))

It turns out that we can see this action via the fundamental group. Recall that the fundamental group given a basepoint 0 is the group of loops up to homotopy equivalence. The fundamental group is also the set of paths (up to homotopy) between the basepoint 0 and its fiber in the universal cover.

The universal cover is usually not a finite cover. To fix this, the key idea is to consider finite coverings and then take a projective limit. This yields an "algebraic" definition of the fundamental group (called the étale fundamental group (?)).

As an example, consider $\mathbb{C}^\times$ with basepoint 0. The universal cover is $\mathbb{C}$ with basepoint 1 via the exponential map and the "classical" π_1 is $\mathbb{Z}$. Let us take the algebraic perspective now. A

finite covering of index (?) n is given by $\mathbb{C}^\times$ with $z \mapsto z^n$, and the fiber is the set of roots of unity μ_n. Therefore our "algebraic" notion of π_1 is the profinite completion

$$\widehat{\mathbb{Z}} = \lim_{\leftarrow} \mu_n$$

There is a clear action of the Galois group on this set. This Galois action is the second hidden symmetry of the variety.

Final remarks: Note that we have a natural wedge $\wedge$ on cohomology classes of X. Suppose we have cohomology classes with representatives being the forms α, β, γ. Suppose that $\alpha \wedge \beta = 0$ and $\beta \wedge \gamma = 0$. Let us compute $\alpha \wedge \beta \wedge \gamma$ in two ways (even though it is clearly zero).

Suppose that $\alpha \wedge \beta = du$ and $\beta \wedge \gamma = dv$ for some forms u, v; recall what it means to be zero in a cohomology group. Then $\alpha \wedge \beta \wedge \gamma = du \wedge \gamma = \pm \alpha \wedge dv$. Rearranging, we find $d(u \wedge \gamma \pm \alpha \wedge v) = 0$, where we used the product rule and the fact that α, γ are closed: $d\alpha = d\gamma = 0$. We have started from α, β, γ and obtained a *new* element of a cohomology group, namely $u \wedge \gamma \pm \alpha \wedge v$. If α, β, γ are p, q, r forms respectively, we have defined a map $H^p \times H^q \times H^r \to H^{p+q+r-1}$. With some heavy machinery one can show that this map must be trivial (?).