MAT325 Study Guide

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Memorize proofs of properties and theorems in Stein–Shakarchi.

Fourier series

Don't forget to adjust Fourier coefficients by the length of the interval/period of the function. In general, on $[0, L]$, our Fourier coefficients are $\hat{f}(n) = \frac{1}{L}\int_0^L f(x)e^{-2\pi i n\frac{x}{L}}dx$.

Summary of convergence results: C^2 implies absolute convergence, C^k yields $o(1/|n|^k)$ decay for $k \in \mathbb{N}$, differentiable at θ_0 implies convergence at θ_0, continuous $\not\Longrightarrow$ convergent (only a.e. by Carleson), α-Holder implies $O(1/|n|^{1+\alpha})$ decay, ($\alpha > \frac{1}{2}$)-Holder implies absolute convergence (Bernstein's Theorem). In another vein, f continuous implies Cesàro and Abel sums converge uniformly; the former yields the density of trig polynomials in $C^0(S^1)$, an analogue to Weierstrass's Theorem.

Remark 0.1. *In Stein–Shakarchi, the definition of an complex inner product space is $(V, \langle\cdot,\cdot\rangle)$ where the inner product is linear in the first entry, Hermitian symmetric, and positive definite (i.e. simply $\langle v, v\rangle \geqslant 0$ for all v). An inner product space is said to be strictly PD if $\langle v, v\rangle = 0 \implies v = 0$. A Hilbert space is (1) SPD and (2) complete. A pre-Hilbert space fails one of the two criteria. In particular, note that $\mathcal{R}[a, b]$ (for any a, b) is a pre-Hilbert space that fails both criteria. In measure theory, this defect of $\mathcal{R}$ is remedied by (1) identifying functions that are equal a.e. and (2) taking a suitable completion. This yields the space L^2, a bona fide Hilbert space.*

Theorem 0.2 (Mean-squared convergence)**.** *For all Riemann integrable f, we have $Sf \to f$ in the L^2 norm.*

Proof. Given N, note $f = S_N f + (f - S_N f)$, where the second term on the RHS lies in the orthogonal complement of $\text{span}\{e_n\}_{|n|\leqslant N}$. The Pythagorean theorem implies that $S_N f$ is the "best approximation" to f among all elements of $\text{span}\{e_n\}_{|n|\leqslant N}$ (i.e. trig polynomials of degree N). To show that the residual goes to 0 in L^2 norm, it remains to find a set of trig polynomials that converge to f in L^2 norm. This is a consequence of Cesàro's theorem, which provides a sequence $\{p_n\}$ converging to f in L^∞ norm. We are done. □

Theorem 0.3 (Parseval). *For Riemann integrable f, we have $\|f\|_{L^2} = \left\|\hat{f}\right\|_{\ell_2}$, i.e. $\int_0^{2\pi} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$. As a result, $\langle f, g\rangle_{L^2} = \langle \hat{f}, \hat{g}\rangle_{\ell_2}$.*

Proof. Corollary of mean-squared convergence of Sf to f. □

Remark 0.4. *Given any orthonormal set $\{e_n\} \subset \mathcal{R}$, we have a weaker form of Parseval known as Bessel's inequality: $\sum_n |\langle f, e_n\rangle|^2 \leqslant \|f\|^2$. When equality holds, we say that $\{e_n\}$ forms a basis for the pre-Hilbert space $\mathcal{R}$. A non-Fourier example of e_n are the Legendre polynomials $\mathcal{L}_n(x) \stackrel{\text{def}}{=} D^n(x^2-1)^n$, the orthonormal basis obtained by "orthogonalizing" the monomials $\{x^n\}$ on $[-1, 1]$. Given N, the Legendre expansion $\sum_{n\leqslant N}\langle f, \mathcal{L}_n\rangle\mathcal{L}_n$ is the "best approximation" to f among all polynomials of degree N, just as the Fourier partial sum $S_N f$ is the "best approximation" to f among all trig polynomials of degree N.*

Remark 0.5. ***Don't forget*** *normalization in Parseval! The standard inner product on $[0, L]$ is $\frac{1}{L}\int_0^L f(x)\overline{g(x)}dx$.*

Corollary 0.6 (Riemann-Lebesgue lemma). *For Riemann integrable f, $\hat{f}(n) \to 0$ as $n \to \infty$.*

Fourier transform on $\mathbb{R}$

Our convention is $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx$. This is well-defined for $f \in M(\mathbb{R})$.

Proposition 0.1 (Properties of FT on $\mathbb{R}$). *Let $f \in S(\mathbb{R})$. Then:*

i) $f(x + h) \mapsto \hat{f}(\xi)e^{2\pi i\xi h}$

ii) $e^{-2\pi ihx}f(x) \mapsto \hat{f}(\xi + h)$

iii) $f(\delta x) \mapsto \delta^{-d}f(\delta^{-1}\xi)$

iv) $\partial^\alpha f(x) \mapsto (2\pi i\xi)^\alpha \hat{f}(\xi)$

v) $(-2\pi ix)^\alpha f(x) \mapsto \partial^\alpha \hat{f}(\xi)$

Proof. We show v). This requires us to show that $\hat{f}(\xi)$ is differentiable and compute its derivative. Write the difference quotient... □

Corollary 0.2. *$S(\mathbb{R})$ is preserved under FT.*

Lemma 0.3 (Product formula). *If $f, g \in S(\mathbb{R})$, then $\int_{\mathbb{R}} \hat{f}(y)g(y)dy = \int_{\mathbb{R}} f(y)\hat{g}(y)dy$.*

Proof. Apply Fubini's Theorem. This is justified, since we have sufficient decay. □

Theorem 0.4 (Fourier inversion). *If $f \in S(\mathbb{R})$, then*

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}d\xi$$

Proof. Note that K_δ satisfies FI, as in fact $\widehat{\widehat{K_\delta}} = K_\delta$. Then apply the product formula to $f = f$, $g = \widehat{K_\delta}$ and use the Good Kernel Theorem to show FI for $x = 0$. Apply to the translated function $f_h(x) = f(x + h)$ to get the general result. □

Theorem 0.5 (Plancherel)**.** *If $f \in S(\mathbb{R})$, then $\|f\|_{L^2} = \left\|\hat{f}\right\|_{L^2}$. As a result, $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$.*

Proof. Apply the Convolution Theorem to $h = f * f^\flat$, where $f^\flat(x) = \overline{f(-x)}$, so that $\widehat{f^\flat}(\xi) = \overline{f(\xi)}$. Then integrate and use FI. **Don't forget** to change bounds when doing the $x \mapsto -x$ u-substitution; write in the bounds! □

Theorem 0.6 (Heisenberg Uncertainty Principle)**.** *Suppose $\psi \in S(\mathbb{R})$ with $\|\psi\|_{L^2} = 1$. Then*

$$\left(\int x^2|\psi(x)|^2dx\right)\left(\int \xi^2|\hat{\psi}(\xi)|^2d\xi\right) \geqslant \frac{1}{16\pi^2}$$

In fact, we have

$$\left(\int (x - x_0)^2|\psi(x)|^2dx\right)\left(\int (\xi - \xi_0)^2|\hat{\psi}(\xi)|^2d\xi\right) \geqslant \frac{1}{16\pi^2}$$

Proof. The second inequality follows from the first by setting $\psi(x) \mapsto e^{-2\pi i\xi_0 x}\phi(x + x_0)$. This function has a tricky FT to compute:

$$\hat{\psi}(\xi) = \widehat{e^{-2\pi i\xi_0 x}\phi_{x_0}}(\xi) = \widehat{\phi_{x_0}}(\xi + \xi_0) = e^{2\pi i(\xi+\xi_0)x_0}\hat{\phi}(\xi + \xi_0)$$

So indeed, $|\psi(x)|^2 = |\phi(x + x_0)|^2$ and $|\hat{\psi}(\xi)|^2 = |\hat{\phi}(\xi + \xi_0)|^2$, allowing for change of variables.

For the first inequality, start with $\|\psi\|_{L^2} = 1$, integrate by parts, take the absolute value, and use C-S. □

Fourier transform on $\mathbb{R}^d$

Our convention is $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi\cdot x}dx$. This is well-defined for $f \in M(\mathbb{R}^d)$.

Proposition 0.1 (Properties of FT on $\mathbb{R}^d$)**.** *Let $f \in S(\mathbb{R}^d)$. Then:*

i) $f(x + h) \mapsto \hat{f}(\xi)e^{2\pi i\xi\cdot h}$

ii) $e^{-2\pi ih\cdot x}f(x) \mapsto \hat{f}(\xi + h)$

iii) $f(\delta x) \mapsto \delta^{-d}f(\delta^{-1}\xi)$

iv) $\partial^\alpha f(x) \mapsto (2\pi i\xi)^\alpha \hat{f}(\xi)$

v) $(-2\pi ix)^\alpha f(x) \mapsto \partial^\alpha \hat{f}(\xi)$

vi) (⋆) $f(Rx) \mapsto \hat{f}(R\xi)$ *for all rotations R*

Proof. We do a few. **Don't forget** the dot products in the exponential!

For i), note $\widehat{f_h}(\xi) = \int f(x+h)e^{-2\pi i\xi x}dx$. Change variables to $y = x+h$, $dy = dx$ to find $\widehat{f_h}(\xi) = \int f(y)e^{-2\pi i\xi\cdot(y-h)}dy = e^{2\pi i\xi\cdot h}\hat{f}(\xi)$.

For iii), note $\widehat{\delta f}(\xi) = \int f(\delta x)e^{-2\pi i\xi\cdot x}dx$. Change variables to $y = \delta x$, $dy = \delta^d dx$ (since the matrix is δI!) to find $\widehat{\delta f}(\xi) = \int f(y)e^{-2\pi i\frac{\xi}{\delta}\cdot y}\cdot\delta^{-d}dy = \delta^{-d}\hat{f}(\delta^{-1}\xi)$.

For iv), we perform IBP on one coordinate, then induct. Note that

$$\widehat{\partial^\alpha f}(\xi) = \int \partial^\alpha f(x)e^{-2\pi i\xi\cdot x}dx = \int\cdots\int \partial^\alpha f(x)e^{-2\pi i\xi_1 x_1}\cdots e^{-2\pi i\xi_d x_d}dx_d\cdots dx_1$$

$$= \int\cdots\int\left[\int \partial_{x_d}^{\alpha_d}\partial^{\alpha'}f(x)e^{-2\pi i\xi_d x_d}dx_d\right]e^{-2\pi i\xi_1 x_1}\cdots e^{-2\pi i\xi_{d-1}x_{d-1}}dx_{d-1}\cdots dx_1$$

$$= \int\cdots\int(2\pi i\xi)^{\alpha_d}\partial^{\alpha'}f(x)e^{-2\pi i\xi'\cdot x'}dx_{d-1}\cdots dx_1 = (2\pi i\xi)^\alpha\hat{f}(\xi)$$

For vi), note that $\widehat{Rf}(\xi) = \int f(Rx)e^{-2\pi i\xi\cdot x}dx$. Change variables to $y = Rx$, $dy = |\det(R)|dx = dx$ to find $\widehat{Rf}(\xi) = \int f(y)e^{-2\pi i\xi\cdot R^{-1}y}dy$. As $\xi\cdot R^{-1}y = R^{-1}(R\xi)\cdot R^{-1}y = R\xi\cdot y$, we get $\widehat{Rf}(\xi) = \hat{f}(R\xi)$. □

Corollary 0.2. *The FT of a radial function is radial.*

Proof. Suppose f is radial, i.e. "$Rf = f$". Then $\hat{f}(R\xi) = \hat{Rf}(\xi) = \hat{f}(\xi)$, as desired. □

Note d-dimensional gaussian is defined as $K_\delta(x) = \delta^{-d/2}e^{-\pi|x|^2/\delta}$, with $\widehat{K_\delta}(\xi) = e^{-\pi\delta|\xi|^2}$.

Fourier inversion and Plancherel for $f \in S(\mathbb{R}^d)$ are proved exactly the same as in $\mathbb{R}$.

Theorem 0.3 (Fourier inversion)**.** *If $f \in S(\mathbb{R}^d)$, then*

$$f(x) = \int_{\mathbb{R}^d}\hat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi$$

Theorem 0.4 (Plancherel)**.** *If $f \in S(\mathbb{R}^d)$, then $\|f\|_{L^2} = \left\|\hat{f}\right\|_{L^2}$. As a result, $\langle f, g\rangle = \langle\hat{f},\hat{g}\rangle$.*

PDEs

Three types: Laplace $0 = \triangle u$, heat $\partial_t u = \triangle u$, wave $\partial_t^2 u = \triangle u$.

Overview: the Laplace equation always involves a Poisson kernel and the heat equation always involves a heat kernel. The heat equation on $\mathbb{R}$ and the wave equation on $\mathbb{R}^d$ have energy identities that imply uniqueness. The Laplace equation ("harmonic functions") have a Maximum Principle that implies uniqueness.

We begin with compact domains (i.e. Fourier series).

1) Wave on $[0, L]$ or $\mathbb{R}$.

Theorem 0.1. *Given initial data $u = f$, $\partial_t u = g$ with $f \in C^2$, $g \in C^1$, the unique C^2 solution to $\partial_t^2 u = \partial_x^2 u$ is given by d'Alembert's equation*

$$u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(\tilde{x})d\tilde{x}$$

Pf: change variables to $\xi = t+x$, $\eta = t-x$ to get $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$. (Recall tricky partial computations!)

Domain $[0, L]$ vs $\mathbb{R}$? Fourier approach? add...

2) Laplace on $\mathbb{D}^2$ (open unit disk).

Recall $P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta+r^2}$ is Poisson kernel.

Theorem 0.2. *Given $f \in C^0(S^1)$, then $u = (P_r * f)(\theta)$ is the unique solution satisfying:*

- *$u \in C^2(\mathbb{D}^2)$ and $\triangle u = 0$ on $\mathbb{D}^2$;*
- *$u \to f$ uniformly as $r \to 1$ (i.e. u C^0-extends to $\overline{\mathbb{D}^2}$).*

Pf: For claim 1, note the series for u may be twice termwise differentiated, so compute. For claim 2, note that $\{P_r\}$ is a good kernel as $r \to 1$. The fact that u C^0-extends only requires a little extra work. For uniqueness... add

Without claim 2, we do not get uniqueness. Ex?... add

3) Heat on S^1 (i.e. $[0, 1]$).

Recall $H_t(x) = \sum e^{-4\pi^2 n^2 t}e^{2\pi i n x}$ is heat kernel of unit circle.

Theorem 0.3. *Given $f \in C^2(S^1)$, then $u = (H_t * f)(x)$ is the unique solution satisfying:*

- *$u \in C^2(S^1)$ with $\partial_t u = \partial_x^2 u$ for all $t > 0$;*
- *$u \to f$ uniformly as $t \to 0$ (i.e. u C^0-extends to $\overline{\mathbb{R}^+ \times S^1}$).*

Pf: For claim 1, compute. For claim 2, note $\{H_t\}$ is a good kernel as $t \to 0$; this requires Poisson summation formula. For uniqueness... add

Without claim 2, we do not get uniqueness. Ex?... add

We now consider non-compact domains (i.e. Fourier transforms).

4) Heat on $\mathbb{R}$.

Since we are working on a non-compact domain, we now require functions in the Schwartz class $S(\mathbb{R})$.

Recall $\mathcal{H}_t(x) = K_{4\pi t}(x)$ is the heat kernel on $\mathbb{R}$, where $K_\delta = \delta^{-1/2} e^{-\pi x^2/\delta}$. Note that the gaussian K_δ is s.t. $\widehat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}$ and $\widehat{\widehat{K}}_\delta = K_\delta$.

Theorem 0.4. *Given $f \in S(\mathbb{R})$, then $u = (\mathcal{H}_t * f)(x)$ is the unique solution satisfying:*

- *$u \in C^2(\mathbb{H}^2)$ with $\partial_t u = \partial_x^2 u$ for all $t > 0$;*
- *$u \to f$ uniformly as $t \to 0$ (i.e. u C^0-extends to $\overline{\mathbb{H}^2}$);*
- *$u \to f$ in the L^2 norm;*
- *($\star$) $u(\cdot, t)$ is uniformly Schwartz.*

Pf: For claim 1, note convolving with $\mathcal{H}_t$ yields smooth function, so compute. For claim 2, note $\{\mathcal{H}_t\}$ is a good kernel as $t \to 0$, as gaussian. For claim 3, use Plancherel. For claim 4, bound the convolution formula for u by splitting the domain into $|y| \geqslant |x|/2$ or $|y| \leqslant |x|/2$, where y is dummy variable. For uniqueness, use claim 4 to justify the "energy identity" $E'(t) \leqslant 0$, where $E(t) = \frac{1}{2}\int_{\mathbb{R}} |u|^2 dx$.

Note that claims 2 and 3 are not equivalent; consider $f_n = \frac{1}{\sqrt{n}} \mathbf{1}_{[0,n]}$, which converges uniformly to $f = 0$ but does not converge to 0 in the L^2 sense. In the other direction, consider $f_n = \mathbf{1}_{[0,1/n]}$, which converges to $f = 0$ in the L^2 sense, but does not converge uniformly to 0.

Without claim 4, we do not get uniqueness... add

5) Laplace on $\mathbb{H}^2$.

Recall $P_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}$ ($\in \mathcal{M}$!) is the Poisson kernel on $\mathbb{H}^2$. Its Fourier transform is given by $\widehat{P}_y(\xi) = e^{-2\pi y|\xi|}$ ($\notin \mathcal{S}$, but $\in \mathcal{M}$!). Fourier inversion and Parseval hold when $f, \hat{f} \in \mathcal{M}$, so they hold for P_y.

We also need a lemma.

Lemma 0.5 (Mean Value Property)**.** *Given u harmonic on an open set $\Omega \subset \mathbb{R}^2$, then for any $(x_0, y_0) \in \Omega$ and disk $D((x_0, y_0), R) \subset \Omega$, we have*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta) d\theta$$

This implies the Maximum Principle for harmonic functions, that if u attains its maximum (minimum) on the interior of Ω, then it must be constant on Ω. I.e. a nonconstant u attains its maximum (minimum) on $\partial\Omega$.

Theorem 0.6. *Given $f \in S(\mathbb{R})$, then $u = (P_y * f)(x)$ is the unique solution satisfying:*

- $u \in C^2(\mathbb{H}^2)$ *with* $\triangle u = 0$ *for all* $y > 0$*;*
- $u \to f$ *uniformly as* $y \to 0$ *(i.e.* u C^0*-extends to* $\overline{\mathbb{H}^2}$*);*
- $u \to f$ *in the* L^2 *norm;*
- *(*$\star$*)* $u \to 0$ *as* $|x| + y \to \infty$ *("*u *vanishes at* ∞*").*

Pf: For claim 1, compute. For claim 2, note $\{P_y\}$ is a good kernel as $y \to 0$. For claim 3, use Plancherel. For claim 4, split into $|x| \geqslant y$ and $|x| \leqslant y$. For uniqueness, use the Maximum Principle.

Without claim 4, we do not get uniqueness; take $u = y$.

Note claim 4 can be thought of as a "boundary condition at ∞".

6) Wave on $\mathbb{R}^d \times \mathbb{R}$.

We begin with "Fourier space" solution on $\mathbb{R}^d \times \mathbb{R}$.

Theorem 0.7. *Given initial data* $u = f$, $\partial_t u = g$ *with* $f, g \in S(\mathbb{R}^d)$*, then the unique solution to* $\partial_t^2 u = \triangle u$ *is given by the inverse Fourier transform*

$$u(x,t) = \int_{\mathbb{R}^d} \left[\hat{f}(\xi) \cos(2\pi|\xi|t) + \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i \xi \cdot x} d\xi$$

$$= \mathcal{F}^*(\hat{f}(\xi) \cos(2\pi|\xi|t) + \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|})$$

Pf: First, note that the function $\hat{f}(\xi) \cos(2\pi|\xi|t) + \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$ is indeed $\in \mathcal{S}(\mathbb{R}^d)$... add. To see that it solves, compute. For uniqueness, show that $E(t) = \int_{\mathbb{R}^d} |\partial_t u|^2 + |\nabla u|^2 dx$ is constant using Plancherel on each term.

We now show the "physical space" solution for $d = 3, 2$.

Lemma 0.8.

$$\frac{1}{4\pi} \int_{S^2} e^{-2\pi i \xi \cdot \gamma} d\sigma(\gamma) = \frac{\sin 2\pi|\xi|}{2\pi|\xi|}$$

Proof. Since both sides are radial, it suffices to show for $\xi = (0, 0, \rho)$. This yields

$$\text{LHS} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} e^{-2\pi i (0,0,\rho) \cdot (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)} \sin\theta d\theta d\phi = \frac{1}{2} \int_0^{\pi} e^{-2\pi i \rho \cos\theta} \sin\theta d\theta$$

$$= \frac{1}{2} \int_{-1}^{1} e^{2\pi i \rho u} du = \frac{\sin 2\pi\rho}{2\pi i \rho} = \frac{\sin 2\pi|\xi|}{2\pi|\xi|}$$

□

Corollary 0.9. $\widehat{M_t f}(\xi) = \hat{f}(\xi) \frac{\sin 2\pi|\xi|t}{2\pi|\xi|t}$.

Theorem 0.10. *For $d = 3$, the above solution is given as $u(x,t) = \partial_t(tM_tf) + tM_tg$, where the spherical mean M_tf is defined as*

$$M_tf(x) = \frac{1}{4\pi}\int_{S^1} f(x - t\gamma)d\sigma(\gamma) = \frac{1}{4\pi t^2}\int_{S(x,t)} f(y)d\sigma(y)$$

Proof. Simply substitute the expressions for $\widehat{M_tf}, \widehat{M_tg}$ into our Fourier space expression for $u(x,t)$. □

Note that the above formula implies the Strong Huygens Principle.

We now descend to $\mathbb{R}^2$.

Theorem 0.11. *For $d = 2$, the solution to the wave equation is given as $u(x,t) = \partial_t(t\tilde{M}_tf)+t\tilde{M}_tg$, where the spherical mean $\tilde{M}_tf$ is defined as*

$$\tilde{M}_tF(x) = \frac{1}{2\pi}\int_{|y|\leqslant 1} \frac{F(x - ty)}{\sqrt{1 - |y|^2}}dy$$

Proof. Given a solution to the wave equation on $\mathbb{R}^2$, extend it to $\tilde{f}, \tilde{g}$ that are constant in x_3. The corresponding $\tilde{u}$ solves the heat equation in $\mathbb{R}^3$. This $\tilde{u}$, when $x_3 = 0$, yields a solution to the heat equation in $\mathbb{R}^2$. (In reality, we need to multiply $\tilde{f}, \tilde{g}$ by a bump function to ensure they lie in $S(\mathbb{R}^3)$.) □

Note that the above formula implies the Weak Huygens Principle.

Other applications

Add: spherical coordinates (which differ from the convenion in $\mathbb{R}^3$)...

Theorem 0.1 (Poisson summation formula)**.** *Given $f \in S(\mathbb{R})$, define the periodizations $F_1(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ and $F_2(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$, which are both continuous 1-periodic functions. Then $F_1(x) = F_2(x)$.*

Proof. Compute the m-th Fourier **series** coefficient of $\sum f(x + n)$ and show that it equals $\hat{f}(m)$. Note that we are integrating over $[0, 1]$,so our Fourier series coefficients are $\int_0^1 F(x)e^{-2\pi imx}dx$. □

Theorem 0.2. *H_t is the (1-)periodization of $\mathcal{H}_t$. As a result, H_t is a good kernel.*

Proof. To see that H_t is the periodization of $\mathcal{H}_t$, note that $\widehat{\mathcal{H}_t}(n) = \widehat{K_{4\pi t}}(n) = e^{-4\pi^2n^2t}$. Thus $\sum_{n\in\mathbb{Z}} \widehat{\mathcal{H}_t}(n)e^{2\pi inx} = \sum_{n\in\mathbb{Z}} e^{-4\pi^2n^2t}e^{2\pi inx}$, which is exactly $H_t(x)$.

To see that H_t is a good kernel, we show that $H_t(x) = \mathcal{H}_t(x) + \mathcal{E}_t(x)$, where $\mathcal{E}_t(x) \leqslant c_1e^{-c_2/t}$ is an error term for $t \in (0, 1]$. This implies that $\int_{|x|\leqslant 1/2} \mathcal{E}_t(x)dx \leqslant c_1e^{-c_2/t} \to 0$ as $t \to 0$, hence

H_t inherits the third property of the definition of a good kernel from $\mathcal{H}_t$. (Here we are thinking of the domain of H_t as $[-1/2, 1/2]$.)

To see this, split the Poisson summation formula into $n = 0$ and $|n| \geqslant 1$:

$$H_t(x) = \mathcal{H}_t(x) + \sum_{|n|\geqslant 1} \mathcal{H}_t(x+n) \leqslant \mathcal{H}_t(x) + Ct^{-1/2} \sum_{|n|\geqslant 1} e^{-(x+n)^2/4t}$$

As $(x+n)^2 \geqslant (-|x| + |n|)^2$, we may bound this above by $\mathcal{H}_t(x) + Ct^{-1/2} \sum_{|n|\geqslant 1} e^{-cn^2/t}$. We want to bound the summation by $O(e^{-c_2/t})$. To do this, note that $n^2/t \geqslant 1/t$ and $n^2/t \geqslant n^2$, thus $e^{-cn^2/t} \leqslant e^{-c/2t} e^{-cn^2/2}$. Our new bound is $\mathcal{H}_t(x) + Ct^{-1/2} e^{-c/2t} \sum_{|n|\geqslant 1} e^{-cn^2/2} \leqslant \mathcal{H}_t(x) + c_2 t^{-c_1/t}$. The result follows. □

Theorem 0.3 (Bessel function in $\mathbb{R}^3$)**.** *Given a radial function $f(x) = f_0(|x|)$ on $\mathbb{R}^3$, its (radial) Fourier transform is given by*

$$\hat{f}(\xi) = F_0(\rho) = 2\rho^{-1} \int_0^\infty \sin(2\pi r\rho) f_0(r) r dr$$

Proof. We would like to use the formula for the FT of the surface element, Lemma 0.8. Since $dx = r^2 d\sigma(\gamma) dr$ (and in general, $dx = r^{d-1} d\sigma(\gamma) dr$), we have

$$F_0(\rho) = \int_{\mathbb{R}^3} f(x) e^{-2\pi i \xi \cdot x} dx = \int_0^\infty r^2 f_0(r) \int_{S^2} e^{-2\pi i \xi \cdot r\gamma} d\sigma(\gamma) dr = \int_0^\infty r^2 f_0(r) 4\pi \cdot \frac{\sin 2\pi r\rho}{2\pi r\rho} dr$$

$$= 2\rho^{-1} \int_0^\infty r f_0(r) \sin(2\pi r\rho) dr$$

Don't forget to change all variables, including $x = r\gamma$, when using spherical! **Don't forget** the $\frac{1}{4\pi}$ in the formula Lemma 0.8! □

Theorem 0.4 (Bessel function in $\mathbb{R}^2$)**.** *If $J_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\rho \sin\theta} e^{-in\theta} d\theta = \widehat{e^{i\rho\sin\theta}}(n)$ (n-th Fourier series coefficient), then given a radial function $f(x) = f_0(|x|)$ on $\mathbb{R}^2$, its (radial) Fourier transform is given by*

$$\hat{f}(\xi) = F_0(\rho) = 2\pi \int_0^\infty J_0(2\pi r\rho) f_0(r) r dr$$

Proof. Note that

$$F_0((0,\rho)) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i \xi \cdot x} dx = \int_0^{2\pi} \int_0^\infty f_0(r) e^{-2\pi i \rho r \sin\theta} r dr d\theta = \int_0^\infty J_0(2\pi r\rho) f_0(r) r dr$$

□

Counterexamples and intuition

To keep in mind for "true and false" questions.

A recurring theme (in the theory of Fourier series and transforms): regularity of f yields decay of $\hat{f}$, and decay of f yields regularity of $\hat{f}$. (The former can be seen with an example: if f has a sharp discontinuity, then it will require high-frequency sinusoids to provide the sudden jump. As a result, $\hat{f}$ will decay slowly.)

Fourier series of continuous function that diverges, built from "symmetry breaking" of $\sum \frac{1}{n} e^{in\theta}$. This yields a "lacunary" Fourier series.

f s.t. $f \in \mathcal{M}$ but $\hat{f} \notin \mathcal{M}$: choose f s.t. f is not Holder (i.e. $1/\log(1/|x|)$), and use the fact that $\hat{f} = O(1/|\xi|^{1+\alpha})$ implies f is α-Holder.

Note that Fourier transform results hold when $f, \hat{f} \in \mathcal{M}$ (see Ch 5, 1.7).

Backwards heat equation: it is strange because it requires initial data that is extremely smooth to solve; specifically, decaying faster than gaussian. Even gaussian "noise" can cause the backwards heat equation to blow up; it is sensitive to high-frequency noise. This sensitivity also implies that the backwards heat equation is "ill-posed": it lacks continuous dependence on initial data (?).

Intuition of "distinguished directions": in each of the PDEs we studied, there was always a distinguished direction (i.e. time t, or $y > 0$ for Laplace). We didn't take the FT including this variable, because that would yield a nonsensical zero solution. What's happening is that aposteriori, we know that the solution will be Schwartz in the non-distinguished directions, but not necessarily in the distinguished directions. Besides, our distinguished directions are not always defined on all of $\mathbb{R}$.

General Fourier analysis: we began to see this with finite Fourier analysis at the end of Stein–Shakarchi.

(Source: `https://terrytao.wordpress.com/2009/04/06/the-fourier-transform/`)

We work on an LCA group G (locally compact abelian). Let μ be a non-trivial Haar measure on G (i.e. translation-invariant Radon measure). Note that the Haar measure is unique up to scaling. Define the additive and multiplicative characters $\xi : G \to \mathbb{R}/\mathbb{Z}$ (frequencies) and $\chi : G \to S^1$ (sinusoids) as continuous homomorphisms. There is a natural correspondence between ξ, χ. We call the group of additive/multiplicative characters the Pontryagin dual $\hat{G}$; analogous to dual space V^* of linear functionals on V. The goal of Fourier analysis is to express $f : G \to \mathbb{C}$, $f \in L^2(\mu)$, as a superposition of multiplicative characters. The Fourier coefficients are $\hat{f}(\xi) = \int f(x) e^{-2\pi i \xi x} dx$. Fourier inversion says $f(x) = \int \hat{f}(\xi) e^{2\pi i \xi x} d\xi$.

Practice problems

1. State and prove the Fourier inversion theorem in the Schwartz class for functions on $\mathbb{R}$. (Prove for gaussians, then consider $f * K_\delta$, use the product formula, and deduce Fourier inversion for $\xi = 0$. Translate to get the result for all ξ.)

2. True or False:

(a) Let $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a global solution of the heat equation $\partial_t u = \triangle u$. Then there exists a non-stationary solution. (True. Note that the backwards heat equation can be solved for super-rapidly decaying initial data; say, compactly supported initial data. Piece together a solution to the forwards and backwards heat equation to find a non-stationary global solution.)

(b) Let $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be a global solution of the wave equation $\partial_t^2 u = \triangle u$. Suppose $\partial_{x_1} u = 0$, $\partial_t \partial_{x_1} u = 0$ for $\{t = 0\} \cap \{r \leqslant R\}$. Then $\partial_{x_1} u = 0$ for $\{r \leqslant R - |t|\}$. (True. Note that $\partial_{x_1} u$ is a solution to the wave equation. Now apply Huygens' Principle.)

(c) Let B denote the closed unit ball in $\mathbb{R}^2$ and let U denote the open unit ball. Let $u : B \to \mathbb{R}$ be a continuous function such that its restriction to U is harmonic. Then u extends to a harmonic function on an open set containing B. (False. Let $u|_{\partial B}$ be continuous but not differentiable. Then $P_r * f$ cannot even extend to a differentiable function outside of B.)

(d) If f is a Schwartz function on $\mathbb{R}^d$, then there exists a Schwartz function u such that $\triangle u = f$. (False. The Fourier transform yields $-4\pi^2|\xi|^2 \hat{u}(\xi, t) = \hat{f}(\xi)$. If $\hat{f}(0) \neq 0$, i.e. $\int f(x)dx \neq 0$, there is no solution $u \in S$! Remark: this is "Poisson's equation" from electromagnetism.)

3. Compare the solutions of the wave equation in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. (See above.)

Extra Stein–Shakarchi practice

Ch 5, Ex 20: For a), take the FT of both sides and apply Poisson summation to $\hat{K}(\xi) = \chi_{[-1/2,1/2]}(\xi)$ (where $K(y) = \frac{\sin \pi y}{\pi y}$). For c), simply take the modulus-squared of a) and note that $K(x-n), K(x-m)$ are orthogonal by Plancherel's isometry; their FTs are distinct exponentials on $[-1/2, 1/2]$.

Ch 6, Ex 5: Write $A = RDR^{-1}$ by spectral theorem. Then $\langle x, Ax\rangle = \langle x, RDR^{-1}x\rangle = \langle R^{-1}x, DR^{-1}x\rangle$. Change variables to $y = R^{-1}x$, $dy = dx$ to find

$$\int_{\mathbb{R}^d} e^{-\pi\langle x,Ax\rangle}dx = \int e^{-\pi\langle y,Dy\rangle}dy = \int e^{-\pi\lambda_1 y_1^2}\cdots e^{-\pi\lambda_d y_d^2}dy_1\cdots dy_d = \lambda_1^{-1/2}\cdots\lambda_d^{-1/2} = \det(A)^{-1/2}$$

Ch 6, Ex 6 (d-dimensional Heisenberg): the RHS changes to $\frac{d^2}{16\pi^2}$. Repeat derivation:

$$1 = \int |\psi|^2 dx = \frac{-1}{d}\sum \int x_j \partial_{x_j}|\psi|^2 dx = -\frac{2}{d}\sum \Re(\langle x_j\psi, \partial_{x_j}\psi\rangle)$$

Taking absolute values and applying C-S,

$$1 \leqslant \frac{2}{d}\sum |\langle x_j\psi, \partial_{x_j}\psi\rangle| \leqslant \frac{2}{d}\sum \|x_j\psi\| \left\|\partial_{x_j}\psi\right\| = \frac{4\pi}{d}\sum \|x_j\psi\| \left\|\xi_j\hat{\psi}\right\|$$

The tricky part is to combine these into the given RHS. To do this, use ordinary C-S:

$$\leqslant \frac{4\pi}{d}(\sum \|x_j\psi\|^2)^{1/2}(\sum \left\|\xi_j\hat{\psi}\right\|^2)^{1/2} = \frac{4\pi}{d}(\sum \int |x|^2|\psi(x)|^2dx)^{1/2}(\int |\xi|^2|\hat{\psi}(\xi)|^2d\xi)^{1/2}$$

Rearranging, we find the desired inequality:

$$\frac{d^2}{16\pi^2} \leqslant (\int |x|^2|\psi(x)|^2dx)(\int |\xi|^2|\hat{\psi}(\xi)|^2d\xi)$$

Ch 6, Ex 7 (d-dimensional heat kernel): note $\mathcal{H}_t^{(d)}(x) = K_{4\pi t}(x)$. Done with Evan.

Ch 6, Ex 8 (d-dimensional Poisson kernel): for a), note that

$$\mathcal{F}(e^{-2\pi|x|})(\xi) = \mathcal{F}^*(e^{-2\pi|x|})(-\xi) = \mathcal{F}^*(\widehat{P_1}(x))(-\xi) = P_1(\xi) = \frac{1}{\pi}\frac{1}{\xi^2+1}$$

Note that

$$\mathcal{F}(\text{RHS})(\xi) = \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}}(u/\pi)^{1/2}\cdot\mathcal{F}((u/\pi)^{-1/2}e^{-\pi x^2/(u/\pi)})(\xi)du = \frac{1}{\pi}\int_0^\infty e^{-u(\xi^2+1)}du = \frac{1}{\pi}\frac{1}{\xi^2+1},$$

as desired.

For b), we apply the subordination principle to the function whose IFT is being evaluated, namely $e^{-2\pi|\xi y|}$. This yields

$$P_y^{(d)}(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{2\pi ix\cdot\xi}\frac{e^{-u}}{\sqrt{\pi u}}e^{-\pi^2|\xi|^2y^2/u}dud\xi$$

$$= \int_{\mathbb{R}}\frac{e^{-u}}{\sqrt{\pi u}}\left[\int_{\mathbb{R}^d} e^{-4\pi^2|\xi|^2\frac{y^2}{4u}}e^{2\pi ix\cdot\xi}d\xi\right]du$$

$$= \int_{\mathbb{R}}\frac{e^{-u}}{\sqrt{\pi u}}\mathcal{H}^{(d)}_{y^2/4u}(x)du = \int_{\mathbb{R}}\frac{e^{-u}}{\sqrt{\pi u}}(\pi y^2/u)^{-d/2}e^{-u|x|^2/y^2}du = \frac{1}{\pi^{(d+1)/2}}\int_{\mathbb{R}}(u/y^2)^{(d-1)/2}e^{-(u/y^2)(y^2+|x|^2)}\frac{1}{y}du$$

$$[v = u/y^2, dv = du/y^2] = \frac{1}{\pi^{(d+1)/2}}\int_{\mathbb{R}} v^{(d-1)/2}e^{-(y^2+|x|^2)v}\cdot ydv$$

$$[w = (y^2+|x|^2)v, dw = (y^2+|x|^2)dv] = \frac{1}{\pi^{(d+1)/2}}\frac{y}{(y^2+|x|^2)^{(d+1)/2}}\int_{\mathbb{R}} w^{(d-1)/2}e^{-w}dw$$

$$= \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}\frac{y}{(y^2+|x|^2)^{(d+1)/2}},$$

as desired.

Ch 6, Ex 10 (alternate proof of $E = \text{const}$): note that

$$E'(t) = \int 2(\partial_t^2 u\partial_t\overline{u} + \partial_t u\partial_t^2\overline{u})dx + \sum\int 2(\partial_t\partial_{x_j}u\partial_{x_j}\overline{u} + \partial_{x_j}u\partial_t\partial_{x_j}\overline{u})dx$$

Note that $\partial_t^2 u = \sum \partial_{x_j}^2 u$ by wave equation. Thus

$$E'(t) = \sum \int 2(\partial_{x_j}^2 u \partial_t \overline{u} + \partial_t u \partial_{x_j}^2 \overline{u}) dx + \sum \int 2(\partial_t \partial_{x_j} u \partial_{x_j} \overline{u} + \partial_{x_j} u \partial_t \partial_{x_j} \overline{u}) dx$$

Apply IBP to the second sum to get

$$E'(t) = \sum \int 2(\partial_{x_j}^2 u \partial_t \overline{u} + \partial_t u \partial_{x_j}^2 \overline{u}) dx - \sum \int 2(\partial_t u \partial_{x_j}^2 \overline{u} + \partial_{x_j}^2 u \partial_t \overline{u}) dx = 0,$$

implying that $E(t) = \text{const}$, as desired.

Ch 6, Ex 12 (dual Radon identity): checking that R^* is indeed the adjoint of R.

$$\int_{\mathbb{R}} \int_{S^2} Rf(t,\gamma) \overline{F(t,\gamma)} d\sigma(\gamma) dt = \int_{\mathbb{R}} \int_{S^2} \int_{\mathbb{R}^2} f(t\gamma + u_1 e_1 + u_2 e_2) \overline{F(t,\gamma)} du_1 du_2 d\sigma(\gamma) dt$$

$$= \int_{\mathbb{R}} \int_{S^2} \int_{\mathbb{R}^2} f(t\gamma + u_1 e_1 + u_2 e_2) \overline{F(t,\gamma)} dt du_1 du_2 d\sigma(\gamma)$$

Change variables to $x = (x_1, x_2, x_3)^T = t\gamma + u_1 e_1 + u_2 e_2 = R(t, u_1, u_2)^T$, $dx = |\det R| dt du_1 du_2 = du_1 du_2 du_3$, where the change of basis matrix is $R = \begin{bmatrix} \gamma & e_1 & e_2 \end{bmatrix} \in O(3)$ (columns are γ, e_1, e_2 wrt the standard basis). Since $t = \gamma \cdot x$, we find

$$= \int_{S^2} \int_{\mathbb{R}^3} f(x) \overline{F(x \cdot \gamma, \gamma)} dx d\sigma(\gamma) = \int_{\mathbb{R}^3} \int_{S^2} f(x) \overline{F(x \cdot \gamma, \gamma)} d\sigma(\gamma) dx = \int_{\mathbb{R}^3} f(x) \overline{R^* F(x)} dx,$$

as desired.