List of DiffGeo Problems (so far)

Sunay Joshi

July 3, 2026

1 HW 1

Exercise 1

Exercise 1.1. *Given a manifold M with charts $\phi_\alpha : U_\alpha \to V_\alpha$, define the following collection of sets $\mathcal{T}$: $A \subset M$ is open iff $\phi_\alpha^{-1}(A \cap V_\alpha)$ is open for all α. Then $\mathcal{T}$ defines a topology on M.*

Exercise 2

Exercise 1.1. *Classify 0-dimensional manifolds.*

Exercise 3

Exercise 1.1. *Let M be a subset of $\mathbb{R}^N$ such that:*

I. For all $p \in M$, there exists a smooth homeo $\phi_p : U_p \to V_p$, where U_p is open in $\mathbb{R}^n$ and V_p is open in the induced topology on M.

II. For all $p \in M$, the differential $d\phi_p|_{\phi_p^{-1}(p)} : \mathbb{R}^n \to \mathbb{R}^N$ is injective.

Then M is an n-dimensional manifold.

Exercise 4

Exercise 1.1. *If $M, \tilde{M}$ are diffeo as manifolds, then their dimensions are equal.*

Exercise 5

Exercise 1.1. *Show the change of coordinates formula: if x^i is one set of coordinates (with chart ϕ) and y^α is another set of coordinates (with chart ψ) around p, then the components of a vector field V satisfy: $V^\alpha = V^i \frac{\partial y^\alpha}{\partial x^i}$. Also, insert the arguments correctly.*

Exercise 6

Exercise 1.1. *Fix local coordinates x^i. Define the Lie bracket of two vector fields as*

$$[X, Y] := (X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}) \frac{\partial}{\partial x^i} \tag{1}$$

Show that this is well-defined, i.e. doesn't depend on choice of coordinates.

Exercise 7

Exercise 1.1. *Show that $\frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial (f \circ \phi)}{\partial x^i}\Big|_{\phi^{-1}(p)}$. (UNOFFICIAL?)*

Exercise 8

Exercise 1.1. *Show that $[X, Y]f = XYf - YXf$.*

Exercise 9

Exercise 1.1. *Given coordinates x^i and y^α, show that the metric transforms as $g_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} g_{ij}$.*

Exercise 10

Exercise 1.1. *Suppose M^n is an immersed submanifold of $(\mathbb{R}^N, \delta)$; that is, suppose M^n is an n-dimensional manifold such that the inclusion map $i : M \to \mathbb{R}^N$ is an immersion. Then the differential $di_p : T_pM \to T_p\mathbb{R}^N$ is injective for all $p \in M$. We define the bilinear form $\tilde{\delta}$ on M^n as follows: for all $p \in M$, we have $\tilde{\delta}_p(v, w) = \delta_p(di_p \cdot v, di_p \cdot w)$. Show that $\tilde{\delta}$ is a metric on M^n (called the induced metric).*

Exercise 11

Exercise 1.1. *Prove the existence of a Riemannian metric using partition of unity.*

Exercise 12

Exercise 1.1. *Given local coordinates x^i, we define the velocity vector field along $\gamma : (a, b) \to M$ to be the map $\gamma' : (a, b) \to TM$ such that $\gamma'(t) = \frac{dx^i}{dt} \cdot \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$. Show that γ' does not depend on the choice of coordinates.*

Exercise 13

Exercise 1.1. *Show that the length of a curve is invariant under reparametrization.*

Exercise 14

Exercise 1.1. *Show that the sum of two Riemannian metrics is again a Riemannian metric. (UNOFFICIAL?)*

Exercise 15

Exercise 1.1. *Show the change of variables formula for integrals on M. Precisely, suppose $f : M \to \mathbb{R}$ is a function that is compactly supported on the intersection $W := V \cap V'$ of two coordinate charts $\phi : U \to V$ and $\psi : U' \to V'$ with local coordinates x^i and y^α. Then show that*

$$\int_{\phi^{-1}(W)} (f \circ \phi)(x^1, \ldots, x^n) \cdot \sqrt{\det g_{ij}} dx^1 \cdots dx^n = \int_{\psi^{-1}(W)} (f \circ \psi)(y^1, \ldots, y^n) \cdot \sqrt{\det g_{\alpha\beta}} dy^1 \cdots dy^n \tag{2}$$

Exercise 16

Exercise 1.1. *Show that in a maximal atlas for a manifold M^n with $n \geqslant 2$, there always exist two charts whose transition map is orientation-reversing. (UNOFFICIAL?)*

Exercise 17

Exercise 1.1. *Show that orientable is equivalent to the existence of a non-vanishing top form. (UNOFFICIAL?)*

Exercise 18

Exercise 1.1. *Prove Sylvester's law of inertia. In addition, check that the signature is an invariant.*

Exercise 19

Exercise 1.1. *If (M^{n+1}, g) is a connected time-oriented Lorentzian manifold, then there are two time-orientations.*

Exercise 20

Exercise 1.1. *Give an example of a non-orientable time-oriented Lorentzian manifold.*

Exercise 21

Exercise 1.1. *Show that in T_pM, the timelike vectors are "inside" the cone, the null vectors are on the boundary of the cone, and the spacelike vectors are "outside" the cone.*

Exercise 22

Exercise 1.1. *If we have global coordinates on our manifold, then any choice of Christoffel symbols yields a connection (via the formula $\nabla_X Y := (X^i \frac{\partial Y^k}{\partial x^i} + X^i Y^j \Gamma^k_{ij}) \frac{\partial}{\partial x^k}$).*

Exercise 23

Exercise 1.1. *Show that any manifold admits a connection via partition of unity.*

Exercise 24

Exercise 1.1. *Part 1.* $\nabla_X Y(p)$ *only depends on* X *through* $X(p)$.

Part 2. $\nabla_X Y(p)$ *only depends on* Y *in an arbitrarily small neighborhood of* p.

2 HW 2

Exercise 1

Exercise 2.1. *If U is open in M, then a connection ∇ on M induces a unique connection on U (?).*

Exercise 2

Exercise 2.1. *Let $\gamma : I \to M$ be an immersed curve (?). Let V be a vector field along γ. Fix $t \in I$ and let $p = \gamma(t)$.*

Part 1: Show that there exists a vector field $\tilde{V}$ coinciding with V near p. (Global or local?)

Part 2: Show that if $\tilde{\tilde{V}}$ is another extension of $\tilde{V}$, then $\nabla_{\dot{\gamma}(t)}\tilde{V} = \nabla_{\dot{\gamma}(t)}\tilde{\tilde{V}}$.

Exercise 3

Exercise 2.1. *Let $\gamma : I \to M$ be a curve, let V be a vector field along γ, and let $p = \gamma(t)$ for some $t \in I$. Let x^i be local coordinates in a neighborhood of p. Show that the formula*

$$\frac{DV}{dt} = (\frac{dV^k}{dt} + \Gamma^k_{ij}\frac{dx^i}{dt}V^j)\frac{\partial}{\partial x^k} \tag{3}$$

for the covariant derivative of V along γ is well-defined under change of coordinates. In particular, determine how the Christoffel symbols transform under change of coordinates.

Exercise 4

Exercise 2.1. *Using the general definition of the covariant derivative (see Exercise 3), show that D/dt along a curve satisfies $\mathbb{R}$-linearity and product rule.*

Exercise 5

Exercise 2.1. *Write up the proof that parallel transport exists along any curve γ.*

Exercise 6

Exercise 2.1. *Suppose that ∇ is compatible with the metric g. Use the general definition of D/dt to show that for any curve γ (including non-immersed curves), compatibility implies that $g(Y,Z) = const$ for any pair of parallel vector fields Y, Z along γ.*

Exercise 7

Exercise 2.1. *Let ∇ be a connection on a manifold with metric g. Suppose that for any pair of parallel vector fields Y, Z along any curve γ, we have that $g(Y,Z) = const$. Show that ∇ is compatible with g.*

Exercise 8

Exercise 2.1. *Let g be a nondegenerate symmetric bilinear form on a vector space V. Show that if $g(v,w) = g(v',w)$ for all $w \in V$, then $v = v'$. (TRIVIAL?)*

Exercise 9

Exercise 2.1. *Show that the matrix $[g_{ij}]$ of a nondegenerate metric is invertible at all points of a semi-Riemannian manifold.*

Exercise 10

Exercise 2.1. *Given local coordinates x^i, recall the formula for the Christoffel symbols Γ_{ij}^m of the Levi-Civita connection:*

$$\Gamma_{ij}^m = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})g^{km} \tag{4}$$

Show that this formula is well-defined under change of coordinates. Also, write down the connection $\nabla(X,Y)$ in the coordinate chart x^i (?).

Exercise 11

Exercise 2.1. *Prove the geometric fundamental theorem of ODEs. (OPTIONAL?)*

Exercise 12

Exercise 2.1. *Suppose that X is a smooth vector field on M. Given $p \in M$, let $\gamma_p : (-T_-(p), T_+(p)) \to M$ denote the maximal integral curve through p. Suppose that $T_+(p) = T_-(p) = \infty$ for all $p \in M$. Let $\varphi_t : M \to M$ denote the flow map for time t, where $\varphi_t(p) = \gamma_p(t)$. Show that $t \mapsto \varphi_t$ is a group homomorphism from $\mathbb{R}$ to M, i.e. $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $t, s \in \mathbb{R}$.*

Exercise 13

Exercise 2.1. *Let $\gamma : I \to M$ be a curve on M. Let $\tilde{\gamma} : I \to TM$ be its lift to the tangent bundle, given by $\tilde{\gamma}(t) = \dot{\gamma}(t)$ (here we abuse notation slightly). Fix a coordinate chart x^α on $V \subset M$, and consider the natural coordinates (x^α, p^α) on the chart $V \times \mathbb{R}^n$ of TM, where (x^α, p^α) maps to the pair (q, v) with $q = (x^1, \ldots, x^n)$ and $v = p^\alpha \frac{\partial}{\partial x^\alpha}$. Show that the tangent vector field along $\tilde{\gamma}$ is given by*

$$\dot{\tilde{\gamma}} = \frac{d^2\gamma^\alpha}{dt^2}\frac{\partial}{\partial p^\alpha} + \frac{d\gamma^\alpha}{dt}\frac{\partial}{\partial x^\alpha} \tag{5}$$

Exercise 14

Exercise 2.1. *Show that $\gamma : I \to M$ is a geodesic iff its lift $\tilde{\gamma} : I \to TM$ satisfies the ODE $(*)$:*

$$\dot{\tilde{\gamma}} = -\Gamma^k_{ij}(\gamma(t))\frac{dx^i}{dt}\frac{dx^j}{dt}\frac{\partial}{\partial p^k} + \frac{dx^k}{dt}\frac{\partial}{\partial x^k} \tag{6}$$

in local coordinates on any natural coordinate chart of TM.

Exercise 15

Exercise 2.1. *Show that the vector field on TM given by*

$$-\Gamma^\mu_{\lambda\nu}(x)p^\lambda p^\nu \frac{\partial}{\partial p^\mu} + p^\mu \frac{\partial}{\partial x^\mu} \tag{7}$$

on any given chart is in fact global; i.e. it is well-defined under a change of coordinates.

Exercise 16

Exercise 2.1. *An integral curve of the above vector field yields a geodesic upon projection from TM to M.*

Exercise 17

Exercise 2.1. *If M is compact, then TM is not compact.*

Exercise 18

Exercise 2.1. *If M is a compact Riemannian manifold, then the subset K_α of TM given by the union of balls*

$$K_\alpha := \bigcup_{p \in M} \{v \in T_pM : g_p(v, v) \leqslant \alpha\} \tag{8}$$

is compact.

Exercise 19

Exercise 2.1. *Let $U \subset TM$ be the set of all $v \in TM$ (abusing notation as usual) such that $T_+(v) > 1$, where $T_+(v)$ denotes the right-endpoint of the interval of definition of the maximal geodesic through p with initial velocity vector v. Given $p \in M$, let $U_p = U \cap T_pM$. Use the geometric fundamental theorem of ODEs to show that the maps $\exp_p : U_p \to M$ and $\exp : U \to M$ are smooth.*

Exercise 20

Exercise 2.1. *Let $(M, \tilde{g})$ be a semi-Riemannian manifold with Levi-Civita connection $\tilde{\nabla}$. Let S be an immersed submanifold of M. Equip S with the semi-Riemannian structure induced by M, and let the induced metric and connection be denoted g, ∇, respectively. Suppose X, Y are vector fields on S, and let $\tilde{X}, \tilde{Y}$ be extensions to M. Show that*

$$\nabla_X Y(p) = \pi \circ \tilde{\nabla}_{\tilde{X}} \tilde{Y}(p) \tag{9}$$

where $\pi : T_pM \to T_pS$ is the orthogonal projection with respect to the inner product $\tilde{g}_p$.

Exercise 21

Exercise 2.1. *Complete the analytic derivation of geodesics on (S^n, δ, ∇) that was started in class, where δ is the induced Euclidean metric on the sphere and ∇ is the Levi-Civita connection.*

Exercise 22

Exercise 2.1. *Suppose $\varphi : (M, g) \to (N, \tilde{g})$ is an isometry of semi-Riemannian manifolds, that is to say $\tilde{g}_{\varphi(p)}(d\varphi_p \cdot v, d\varphi_p \cdot w) = g_p(v, w)$ for all $v, w \in T_pM$ and for all $p \in M$. Show that if $\gamma : I \to M$ is a geodesic, then the image $\varphi \circ \gamma$ of γ under the isometry φ is again a geodesic.*

Exercise 23

Exercise 2.1. *Show that isometries preserve the lengths of curves, as well as the angles between pairs of curves.*

Exercise 24

Exercise 2.1. *Show that straight lines minimize length in $(\mathbb{R}^n, \delta)$.*

3 HW 3

Exercise 1

Exercise 3.1. *Let $\varphi : A \to M$ be a parametrized surface. By abuse of notation, let $\frac{\partial}{\partial t}, \frac{\partial}{\partial s}$ denote the pushforward by φ of the coordinate vector fields $\frac{\partial}{\partial t}, \frac{\partial}{\partial s}$ on A. Show that $\frac{D}{\partial s}\frac{\partial}{\partial t} = \frac{D}{\partial t}\frac{\partial}{\partial s}$.*

Exercise 2

Exercise 3.1. *Radial geodesics are orthogonal to geodesic spheres.*

Exercise 3

Exercise 3.1. *Let (M, g) be Riemannian. Let $p \in M$. Let $N = \exp_p(B_\epsilon(0))$ be a normal ball. Let $q \in N$. Consider the radial geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$. Then γ minimizes length among all piecewise smooth curves $\tilde{\gamma} : [0, 1] \to M$.*

Exercise 4

Exercise 3.1. *Show that in geodesic Cartesian coordinates around p, the metric is diagonal at p and the Christoffel symbols vanish at p.*

Exercise 5

Exercise 3.1. *Let (M, g) be Riemannian. Let $p \in M$. Then there exists a totally normal neighborhood of p in the following sense: there exists $\delta > 0$ and $U \ni p$ such that $U \subseteq \exp_q(B_\delta(0))$ for any $q \in U$.*

Exercise 6

Exercise 3.1. *Show that the Riemann curvature tensor $R(X, Y)Z$ is $C^\infty(M)$-linear in its third entry Z.*

Exercise 7

Exercise 3.1. *Recall that the sectional curvature of a 2-plane $\sigma \subseteq T_pM$ at a point $p \in M$ is given by*

$$K(X,Y) = \frac{(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X,Y\rangle^2} \tag{10}$$

where X, Y is a given basis of σ. Show that sectional curvature $K(\sigma)$ is independent of the choice of basis X, Y.

Exercise 8

Exercise 3.1. *Recall that the Ricci curvature of X, Y at a point $p \in M$ is given by the trace*

$$Ric(X,Y) = \frac{1}{n-1}\sum_{i=1}^{n}(X,e_i,Y,e_i) \tag{11}$$

where $\{e_1,\ldots,e_n\}$ is a given orthonormal basis for T_pM. Show that the Ricci curvature is independent of the choice of orthonormal basis for T_pM.

4 HW 4

Exercise 1

Exercise 4.1. *A manifold M is isotropic at p if $K(p,\sigma) = K(p,\tilde{\sigma})$ for all pairs of 2-planes $\sigma, \tilde{\sigma}$ inside of T_pM. Prove Schur's lemma: let $\dim M \geqslant 3$, and suppose M is isotropic at p for all p. Then $K(p,\sigma) = K_0$ is a constant for all p and for all $\sigma \subset T_pM$.*

Exercise 2

Exercise 4.1. *Suppose M is has constant sectional curvature K_0. Suppose γ is a geodesic that is parametrized by arclength. Suppose J is a Jacobi field along γ such that $\langle J, \gamma' \rangle = 0$. Then $R(\gamma', J)\gamma' = K_0 J$.*

Exercise 3

Exercise 4.1. *Let $\gamma : [0,1] \to M$ be a geodesic. Let J be a Jacobi field along γ with $J(0) = 0$. Then*

$$J(t) = (d\exp_p)_{t\gamma'(0)} \cdot tJ'(0) \tag{12}$$

Exercise 4

Exercise 4.1. *Let $\gamma : [0,a] \to M$ be a geodesic parametrized by arclength, with $\gamma(0) = p$ and $\gamma'(0) = v$. Let J be the unique Jacobi field along γ with $J(0) = 0$ and $J'(0) = w$, where $\langle v, w \rangle = 0$. Recall that*

$$|J(t)|^2 = t^2 - \frac{1}{3}K_p(\sigma)t^4 + \mathcal{R}(t) \tag{13}$$

where $\sigma = span\{v, w\}$ and $\mathcal{R} = o(t^4)$. Show that

$$|J(t)| = t - \frac{1}{6}K_p(\sigma)t^3 + \tilde{\mathcal{R}}(t) \tag{14}$$

where $\tilde{\mathcal{R}} = o(t^3)$.

Exercise 5

Exercise 4.1. *In the context of the previous exercise, show that*

$$\left.\frac{d}{dt}\right|_{t=0}(\gamma', J, \gamma', J') = 0 \tag{15}$$

Exercise 6

Exercise 4.1. *Conjugate points are well-defined under reparametrization of the geodesic γ.*

Exercise 7

Exercise 4.1. *The unit n-sphere S^n has constant sectional curvature equal to $K_0 = 1$.*

Exercise 8

Exercise 4.1. *Suppose $\gamma(t_0)$ is conjugate to $\gamma(0)$ along the geodesic $\gamma : [0,1] \to M$ for some $t_0 > 0$. Then the multiplicity of $\gamma(t_0)$ is* $\dim \ker(d\exp_p)_{t_0\gamma'(0)}$.

Exercise 9

Exercise 4.1.

1. *Let $\gamma : (0,\infty) \to \mathbb{R}^2$ be a curve such that $\gamma : (0,a) \to \mathbb{R}^2$ is a curve of constant curvature in the yz-plane with $\lim_{t\downarrow 0}\gamma(t) = 0$ and $\lim_{t\downarrow 0}\gamma'(t)$ not parallel to the y-axis extended linearly for $t \geqslant a$, and let S be the surface of revolution obtained by revolving γ around the z-axis. Show that S is inextendible.*
2. *Show that the α-degree cone ($\alpha < \pi/2$) contained in $\{(x,y,z) : z > 0\}$ is inextendible as a Riemannian submanifold of $\mathbb{R}^3$. (Here α is the angle between a meridian and the z-axis.)*
3. *Consider a time-like α-degree cone (i.e. $\alpha < \pi/4$) in the Minkowski space $\mathbb{R}^{2+1}$. This is a Lorentzian $1+1$-dimensional submanifold of $\mathbb{R}^{2+1}$. Is it extendible? (SKIPPED)*

Exercise 10

Exercise 4.1. *M is called homogeneous if for any pair of points p, q, there exists an isometry sending $p \mapsto q$. Show that homogeneous Riemannian manifolds are complete. What about semi-Riemannian manifolds?*

5 HW 5

Exercise 1

Exercise 5.1. *Recall that in the proof of Hadamard's Theorem, we define the metric $\tilde{g}$ on T_pM such that $\tilde{g}$ is the pullback of g via $\exp_p$. Suppose we instead define $\tilde{g}$ to be the metric induced by the inner product g on T_pM. Show that in general, the coercivity assumption $\tilde{g}_w(v,v) \geqslant g_q(df_w \cdot v, df_w \cdot v)$ may fail to hold, where $f = \exp_p$, $w \in T_pM$, and $v \in T_w(T_pM)$. (In the case of $\tilde{g} = \exp_p^* g$, this is an equality.)*

Exercise 2

Exercise 5.1. *Show that local isometries preserve the curvature tensor. (SKIP?)*

Exercise 3

Exercise 5.1. *Show that $\mathbb{H}^n$ is a regular submanifold of Minkowski space $\mathbb{R}^{n+1}$. (TRIVIAL?)*

Exercise 4

Exercise 5.1. *Show that $E(n)$, the group of isometries of Euclidean space $\mathbb{R}^n$, is generated by $O(n)$ and translations.*

Exercise 5

Exercise 5.1. *Show that the Poincaré group, the group of isometries of Minkowski space $\mathbb{R}^{n+1}$, is generated by $O(n,1)$ and translations. (Recall that $O(p,q)$ consists of all A such that $A^T B_{p,q} A = B_{p,q}$, where $B_{p,q} = diag(-1,\dots,-1,1,\dots,1)$, where there are q -1's.)*

Exercise 6

Exercise 5.1. *Show that $O(n)$ is generated by $-I$ and, for each pair $1 \leqslant i < j \leqslant n$, the matrix that rotates the (i,j) plane by θ counterclockwise.*

Exercise 7

Exercise 5.1. *Show that $SO(1,1)$ is generated by $-I$ and, for $\alpha \in \mathbb{R}$, the Lorentz boosts*

$$\begin{bmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{bmatrix} \tag{16}$$

Using this fact, show that for $n \geqslant 2$, $SO(n,1)$ is generated by:

- *for $\alpha \in \mathbb{R}$ and $1 \leqslant i \leqslant n$, the Lorentz boost corresponding to α in the (x^0, x^i)-plane;*
- *the negation map in the (x^0, x^i)-plane;*
- *rotations of the hyperplane $x^0 = 0$ ("space-rotations").*

(Recall that Minkowski space uses zero-indexed coordinates $(x^0, x^1, \ldots, x^n)$.)

Exercise 8

Exercise 5.1. *Show the following form of isotropy for Minkowski space: given $p, q \in \mathbb{R}^{n+1}$, $v \in T_p\mathbb{R}^{n+1}$, $w \in T_q\mathbb{R}^{n+1}$, with $|v| = |w|$, show that there exists $\varphi \in O(n,1)$ such that $\varphi(p) = \varphi(q)$ and $d\varphi_p \cdot v = w$.*

Exercise 9

Exercise 5.1. *In Euclidean space, any orthonormal basis at p can be taken to any orthonormal basis at q via an isometry of $\mathbb{R}^n$. If the two bases have the same orientation, this isometry can be taken to be a composition of a translation and a rotation in $SO(n)$.*

Exercise 10

Exercise 5.1. *A pseudo-orthonormal basis in Minkowski space consists of a timelike vector of length -1 and n spacelike vectors of length 1, all of which are orthogonal. Show that any pseudo-orthonormal basis at p can be taken to any pseudo-orthonormal basis at q via a translation and an element in $O(n,1)$.*

Exercise 11

Exercise 5.1. *Show that as a Riemannian manifold, $\mathbb{H}^n$ has constant negative sectional curvature $K = -1$.*

Exercise 12

Exercise 5.1. *Let M be a complete simply connected Riemannian manifold of constant sectional curvature $+1$. If $n := \dim M$ is even, show that M must be isometric to S^n or $\mathbb{RP}^n$. (In class we showed that M is necessarily isometric to some quotient of S^n.)*

Exercise 13

Exercise 5.1. *Recall the following construction of the Poincaré disk model of hyperbolic space. Consider $\mathbb{H}^n$ as the hyperboloid $\{-(x^0)^2 + (x^1)^2 + \ldots + (x^n)^2 = -1\}$ in Minkowski space $\mathbb{R}^{n+1}$. Consider the map $\varphi : \mathbb{H}^n \to B^{n-1}$, where B^{n-1} denotes the ball $\{(x^1)^2 + \ldots + (x^n)^2 \leqslant 1\}$, given by: for each $p \in \mathbb{H}^n$, draw the line through $(0, \ldots, 0, -1)$ and p, and let it intersect the hyperplane $x^0 = 0$ at $\varphi(p)$. Show that φ is a smooth map, and show that the pushfoward of the induced metric is given by*

$$\varphi_* g = \frac{4((dx^1)^2 + \ldots + (dx^n)^2)}{(1 - ((x^1)^2 + \ldots + (x^n)^2))^2} \tag{17}$$

Exercise 14

Exercise 5.1. *Consider the following upper half-space model: equip the set $\{(y^1, \ldots, y^n) : y^n > 0\} \subseteq \mathbb{R}^n$ with the metric*

$$g = \frac{1}{y_n^2}((dy^1)^2 + \ldots + (dy_n)^2) \tag{18}$$

Show that this manifold is isometric to $\mathbb{H}^n$.

6 HW 6

Exercise 1

Exercise 6.1. *Show that $O_+(n,1)$ acts on the sphere S^{n-1} at infinity by conformal mappings. (Note that the sphere at infinity can be identified with the intersection of the space-like hyperplane $\{x^0 = 0\}$ with the future lightcone in Minkowski space $\mathbb{R}^{n+1}$.)*

Exercise 2

Exercise 6.2. *Consider n-dimensional de Sitter space $(\mathcal{S}, g)$, the Lorentzian submanifold of Minkowski space $\mathbb{R}^{n+1}$ given by the hyperboloid $\{-x_0^2 + x_1^2 + \ldots + x_n^2 = \alpha^2\}$. We showed in class that $\mathcal{S}$ has constant sectional curvature K. Compute K as a function of α.*

Exercise 3

Exercise 6.3. *In the first part of this problem, we consider an alternative model of de Sitter space. Consider the subset $D = \{(t, r, x) \in \mathbb{R} \times (0, \alpha) \times S^{n-2}\}$ of $\mathbb{R}^{(n-1)+1}$. Geometrically this is a cylinder with its axis of revolution removed. Consider the metric*

$$g = -(1 - \frac{r^2}{\alpha^2})dt^2 + (1 - \frac{r^2}{\alpha^2})^{-1}dr^2 + r^2 d\gamma^2_{S^{n-2}} \tag{19}$$

on D.

(1) Show that g is a Lorentzian metric on D.

(2) Show that D trivially extends to $r = 0$.

(3) Show that D is isometric to an open subset of de Sitter space $\mathcal{S}$.

(4) What happens as $r \to \alpha$?

Next, we define anti de Sitter space $\tilde{\mathcal{S}}$ – a Lorentzian manifold with negative constant sectional curvature. (With the introduction of anti de Sitter space, we now have model spacetimes with positive, zero, and negative constant sectional curvature.)

We consider the space $\mathbb{R}^{(n-1)+2}$ with coordinates $(t_1, t_2, x_1, \ldots, x_{n-1})$ and metric $-dt_1^2 - dt_2^2 + dx_1^2 + \ldots + dx_{n-1}^2$. (This is not Minkowski space, as it has two time dimensions.) We define anti de Sitter space to be the hyperboloid $-t_1^2 - t_2^2 + x_1^2 + \ldots + x_{n-1}^2 = -\alpha^2$ with the induced metric from $\mathbb{R}^{(n-1)+2}$.

We now provide an alternative model of anti de Sitter space. Similar to the above, consider the metric

$$g = -(1 + \frac{r^2}{\alpha^2})dt^2 + (1 + \frac{r^2}{\alpha^2})^{-1}dr^2 + r^2 d\gamma^2_{S^{n-2}} \tag{20}$$

on the set $D' = \mathbb{R} \times (0, 2\pi\alpha) \times S^{n-2}$. *(In other words, we simply change the sign in front of* α^2 *and extend the range of* r*.)*

(5) Show that D' *is isometric to an open subset of anti de Sitter space* $\tilde{\mathcal{S}}$.

(6) Show that Minkowski, de Sitter, and anti de Sitter space are all geodesically complete. (Note that we cannot appeal to Hopf–Rinow, which only holds for Riemannian manifolds!)

Exercise 4

Exercise 6.4. *Recall the definition of a variation from class: given a piecewise smooth curve* $\gamma : [0, a] \to M$*, a variation is a map* $\tilde{\gamma} : (-\epsilon, \epsilon) \times [0, a] \to M$ *satisfying* $\tilde{\gamma}(0, t) = \gamma(t)$*,* $\tilde{\gamma}$ *is continuous, and there exist* $t_1, \ldots, t_k \in [0, a]$ *such that the restriction of the variation* $\tilde{\gamma}|_{(-\epsilon,\epsilon)\times[t_i,t_{i+1}]}$ *is smooth for all* $1 \leqslant i \leqslant k - 1$.

Show that (when $\tilde{\gamma}$ *is thought of as a parametrized surface) the vector field* $\frac{\partial \tilde{\gamma}}{\partial s}$ *is piecewise smooth along each* $\tilde{\gamma}_s$.