DiffGeo notes

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Professor Dafermos: global theorems in Riemannian geometry are typically of the form: "a local geometric assumption and a weak topological assumption combine to yield a strong global geometric property." In Lorentzian geometry, things are the other way around...

Given ONB $\{e_i\}$ at p, $Ric_p(x,y) = \frac{1}{n-1}\sum(x, e_i, y, e_i)$ in do Carmo convention. (Other authors omit the $\frac{1}{n-1}$.) Equivalently, Ricci is the trace of linear map $z \mapsto Ric(x,z)y$. In particular $Ric_p(v) = \frac{1}{n-1}\sum(v, e_i, v, e_i)$. Recall $K(x,y) = \frac{(x,y,x,y)}{|x\wedge y|^2}$ is the sectional curvature of $\sigma = span\{x,y\}$, i.e. the gauss curvature of $\exp(\sigma)$ as an embedded submanifold. So Ricci is an average of sectional curvatures. It is clear that Ric is a 2-tensor. In coordinates x_i, Ric can be obtained by contracting the second and fourth components of the Riemann curvature 4-tensor: $Ric_{ik} = \sum_{j,l} R_{ijkl}g^{jl}$.

1 Hadamard

Complete + nonpositive sectional curvature + s.c. implies diffeo to $\mathbb{R}^n$. Further, the diffeo is given by the exponential map $\exp_p : T_pM \to M$ (which in this case is defined on all of T_pM).

add...

2 Model spaces

Professor Dafermos: "There is no end to the study of S^n, $\mathbb{H}^n$, and their various quotients. A good geometer should be able to navigate these with ease."

Quotient by the action of a group G of isometries that act properly discontinuously (else imagine quotienting $\mathbb{R}^2$ by group generated by reflection about the x-axis)...

Quotients of S^n: S^n (s.c. if $n \geqslant 2$), double covers RP^n (orientable if n odd)... (When $n = 3$, things are interesting: S^3 is homeo to $SU(2)$ and RP^3 is homeo to $SO(3)$.)

Quotients of $\mathbb{R}^n$: $\mathbb{R}^n$, if we quotient by the action of a lattice Γ by translations, then we obtain a torus $\mathbb{R}^n/\Gamma$, the fundamental domain of a Γ-periodic function.

Quotients of $\mathbb{H}^n$: when $n = 2$, using the upper halfplane model, if we quotient by the action of $PSL_2(\mathbb{Z})$ by Mobius transformations, then we obtain the fundamental domain of a modular form (think an Eisenstein series defined by an integer lattice)...

add...

3 Bonnet-Myers

Bonnet-Myers: Suppose (M, g) is a complete Riemannian manifold. Suppose $Ric_p(v) \geqslant \frac{1}{r^2} > 0$ for all p, v with $|v| = 1$. Then $diam(M) \leqslant \pi r$. Hence M compact (?).

(Think sphere: all sectional curvatures are $\frac{1}{r^2}$, so $Ric \equiv \frac{1}{r^2}$. Diameter of the sphere is πr.)

We want to show that for all p, q, there exists a curve γ joining p to q with length $\leqslant \pi r$.

Since M is complete, by Hopf-Rinow there exists a length-minimizing geodesic γ connecting p to q. Let $\gamma : [0, 1] \to M$ have length ℓ. We claim $\ell \leqslant \pi r$. We apply the "second derivative test" to the *energy* functional, which has a local minimum at γ. This reads $E''(0) \geqslant 0$ for any variation of γ.

The trick is to construct $n - 1$ variations, each moving in an independent direction.

Parallel transport an ON frame $\{e_i(t)\}$ along γ, where e_1 is chosen parallel to γ', i.e. $e_1(t) = \frac{1}{\ell}\gamma'(t)$. (Recall the speed is ℓ.)

For each j in $2, \ldots, n$, we construct a variation in the direction e_j. Let $V_j(t) = \sin(\pi t)e_j(t)$ be the variational vector field (which when integrated yields the actual variation). This is a proper variation because $V_j(0) = V_j(1) = 0$; the sine wave makes a half-period. Recall if γ is a geodesic, then any proper variation is automatically a critical point of the energy: $E'(0) = 0$. So it makes sense to compute $E''(0)$ via the second variation formula. For a proper variation, the quantity $E_j''(0)$ can be expressed in terms of the index form as

$$E_j''(0) = -2 \int_0^1 (\langle V'', V\rangle - (\gamma', V, \gamma', V))dt \tag{1}$$

Here we use the shorthand V'' for the second covariant derivative $\frac{D^2V}{dt^2}$. (This can be checked with a lot of tedious calculation.) We see the sectional curvature of $span\{\gamma', V\}$ sitting inside the integrand. If we average over j, a Ricci term should pop out.

But we must first massage the integrand. Note $V_j = \sin(\pi t)e_j$. Since e_j is parallel, $V_j'' = -\pi^2 V_j$, so that $\langle V_j'', V_j\rangle = -\pi^2 \sin^2(\pi t)$. Next, since $\gamma' = \ell e_1$, multilinearity of R yields $(\gamma', V, \gamma', V) = \ell^2 \sin^2(\pi t)(e_1, e_j, e_1, e_j)$.

Now we average over j in $2, \ldots, n$. Since by definition $Ric_p(e_1) = \frac{1}{n-1}\sum(e_1, e_j, e_1, e_j)$, we obtain

$$\frac{1}{n-1}\sum E_j''(0) = -2 \int_0^1 (-\pi^2 \sin^2(\pi t) - \ell^2 \sin^2(\pi t) \cdot Ric_{\gamma(t)}(e_1))dt \tag{2}$$

Since γ is length-minimizing, the RHS is nonpositive, so that

$$\int_0^1 (\pi^2 \sin^2(\pi t) - \ell^2 \sin^2(\pi t) \cdot Ric_{\gamma(t)}(e_1)) \geqslant 0 \tag{3}$$

We isolate ℓ, the quantity of interest. Beware: $Ric_{\gamma(t)}(e_1(t))$ depends on t. But luckily, it is bounded below by $\frac{1}{r^2}$. So

$$\pi^2 r^2 = \frac{\int_0^1 \pi^2 \sin^2(\pi t)}{\int_0^1 \frac{1}{r^2} \sin^2(\pi t)} \geqslant \ell^2 \tag{4}$$

as desired. ■

So the length bound is really coming out of the inequality

$$L(\gamma)^2 \leqslant \frac{\int_0^1 \pi^2 \sin^2(\pi t)}{\int_0^1 Ric(e_1) \sin^2(\pi t)} \tag{5}$$

for any length-minimizing geodesic γ. The function $f(t) = \sin(\pi t)$ is specially chosen. Maybe others work too.

How can we remember the second variation formula?

$$E''(0) = -2 \int_0^1 (\langle V'', V \rangle - (\gamma', V, \gamma', V)) \tag{6}$$

Let's consider the Euclidean setting, where the second term is absent and where covariant derivatives are vanilla derivatives. Let $\gamma : [0, 1] \to \mathbb{R}^n$ be a geodesic (i.e. straight line) and V a proper variational vector field. By definition $E(s) = \int_0^1 \langle \partial_t \gamma, \partial_t \gamma \rangle$ (abusing the notation "γ")

Differentiate: $E'(s) = \int_0^1 2\langle \partial_s \partial_t \gamma, \partial_t \gamma \rangle$

Commute partials without picking up curvature, then do product rule trick:

$E'(s) = \int_0^1 2(\partial_t \langle \partial_s \gamma, \partial_t \gamma \rangle - \langle \partial_s \gamma, \partial_t^2 \gamma \rangle)$

FTC + proper: $-\frac{1}{2} E'(s) = \int_0^1 \langle \partial_s \gamma, \partial_t^2 \gamma \rangle$

Differentiate once more and evaluate at $s = 0$ (i.e. our original γ): $-\frac{1}{2} E''(0) = \int_0^1 \langle \partial_s^2 \gamma, \partial_t^2 \gamma \rangle + \int_0^1 \langle \partial_s \gamma, \partial_s \partial_t^2 \gamma \rangle$

The first term is $\langle V', \gamma'' \rangle = 0$ since γ is a geodesic. Commuting partials without picking up curvature, the second term is $\langle V, V'' \rangle$, so that $-\frac{1}{2} E''(0) = \int_0^1 \langle V, V'' \rangle$. By IBP, we also have $\frac{1}{2} E''(0) = \int_0^1 |V'|^2$.

To include the curvature terms, use the relation $\partial_t \partial_s - \partial_s \partial_t = R(\partial_s, \partial_t)$, which comes from the definition of the Riemann curvature tensor. (Here each partial should be interpreted properly.)

4 Synge-Weinstein

Our goal is the following relation between positive sectional curvature and simply-connectedness/orientability.

Corollary (Synge's theorem): if M is compact Riemannian manifold with positive sectional curvature, then

- If n is even and M is orientable, then M is simply-connected
- If n is odd, then M is orientable

The theorem statement is a bit clunky, since it arrives at different conclusions based on the parity of n. Let's consider some examples.

S^n: when n even (think S^2), Synge implies that S^n is simply-connected. However, when n odd, we can't guarantee simply-connected; RP^3 has fundamental group $\mathbb{Z}/2$, as it is double covered by S^3.

RP^n: recall that this is the space of lines through the origin in $\mathbb{R}^{n+1}$, or equivalently the result of identifying antipodal points on S^n. Recall that RP^2 is non-orientable (think of a gluing diagram). Synge implies that RP^3 is orientable (in fact, RP^n for any odd n).

How do we prove Synge? Well, at least in the even case, it seems useful to study the universal cover $\tilde{M}$ of M. If we can show the group of deck transformations is trivial, then M is simply-connected.

Recall that $\tilde{M}$ is defined purely topologically. A natural question is whether $\tilde{M}$ admits a differentiable structure. The answer is yes. There exists a "Riemannian covering metric" $\tilde{g}$ on $\tilde{M}$ such that the covering map $\pi : \tilde{M} \to M$ is a local isometry. It's defined in the obvious way, by pulling back g by π. Thus all completeness/curvature assumptions on (M, g) lift to $(\tilde{M}, \tilde{g})$.

A diffeo f of $\tilde{M}$ is a deck transformation iff $\pi \circ f = \pi$. (This is with respect to M, of course.) These deck transformations are isometries?... It turns out that the group of deck transformations is isomorphic to $\pi_1(M)$. (Think of $\pi : \mathbb{R}^2 \to \mathbb{T}^2$...)

To prove Synge's theorem when n is even, we want to study the deck transformations of $(\tilde{M}, \tilde{g})$. If $\pi_1(M) \neq 0$, then there exists a deck transformation $\phi \neq e$ without fixed points...

We therefore ask when a manifold with positive sectional curvature has an isometry with no fixed points...

Synge-Weinstein: Let (M, g) be a compact Riemannian manifold with positive sectional curvature. Let f be an isometry of M. If n even, assume f preserves orientation. If n odd, assume f reverses orientation. Then f has a fixed point.

Let's see how Synge-Weinstein implies Synge. Since M is compact, $K \geqslant \varepsilon > 0$ for some constant ε. Then $(\tilde{M}, \tilde{g})$ is complete and inherits $K \geqslant \varepsilon$. By Bonnet-Myers, $(\tilde{M}, \tilde{g})$ is compact.

Orientability... If n even, we can apply Synge-Weinstein to $(\tilde{M}, \tilde{g})$ to deduce that all isometries of $\tilde{M}$ have fixed points. Thus there cannot exist any non-identity deck transformations of $\tilde{M}$. Equivalently, $\pi_1(M) = 0$.

n odd...

Read backwards, Synge tells us what metrics we can put on a given topological space. There is no metric on $\mathbb{T}^2$ with positive sectional curvature. You can ask the same about $S^2 \times S^2$.

5 Rauch comparison

Index lemma (only one trick), Rauch comparison, length comparison (duality), Sphere theorem..

6 Classical Lie groups

add...

7 Curvature dimension conditions

Ricci lower bound equivalent to Bakry-Emery log-Sobolev inequality (Sara discussed in the case of the generator being the Laplacian), rate of convergence of Markov process to equilibrium measure (i.e. the law μ_t of a solution to SDE converges to some μ), analogy with discrete Markov chains, mixing time, MCMC, Langevin MC requires further discretization in time (which must be done carefully), Hamiltonian MC...

8 Wasserstein geometry

From discussion with Dwip, restrict to submanifold $\mathcal{P}_2$, the W_2 distance on the manifold allows us to interpolate between μ and ν via a geodesic, geodesic convexity, metric?...