Advanced Calculus

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July 3, 2026

Below, we collect some useful results from advanced calculus. (This was intended to make the technical details of Fourier analysis more transparent.)

Lebesgue criterion:

Recall that the standard domain of Riemann integration is a closed cell $\Delta \subset \mathbb{R}^n$.

Theorem 0.1 (Lebesgue criterion)**.** *f is integrable on $\Delta \subset \mathbb{R}^n$ iff it is discontinuous on a set of measure 0 (i.e. it is continuous a.e.).*

Fubini's theorem:

Source: Gunning, Chapter 6-2

Theorem 0.2. *Suppose $f(x_1, x_2)$ is integrable on $\Delta = \Delta_1 \times \Delta_2 \subset \mathbb{R}^n$, where $\Delta_1 \subset \mathbb{R}^{n_1}$, $\Delta_2 \subset \mathbb{R}^{n_2}$. Assume that $f(x_1, x_2)$ is integrable with respect to x_1. Then $\int_{x_1 \in \Delta_1} f(x_1, x_2) dx_1$ is integrable with respect to x_2 and we have the equality*

$$\int_\Delta f dx = \int_{x_2 \in \Delta_2} \int_{x_1 \in \Delta_1} f(x_1, x_2) dx_1 dx_2$$

Corollary 0.3. *If $f(x_1, x_2)$ is continuous on $\Delta = \Delta_1 \times \Delta_2 \subset \mathbb{R}^n$, then*

$$\int_\Delta f dx = \int_{x_1 \in \Delta_2} \int_{x_1 \in \Delta_1} f(x_1, x_2) dx_1 dx_2 = \int_{x_1 \in \Delta_1} \int_{x_2 \in \Delta_2} f(x_1, x_2) dx_2 dx_1$$

Remark 0.4. *Note that if f is integrable on Δ, it does not follow that for fixed x_2, $f(x_1, x_2)$ is integrable with respect to x_1. This is why we assume that $f(x_1, x_2)$ is integrable with respect to x_1.*

Remark 0.5. *Practically, when using Fubini for a function f that is simply integrable, be sure to check that $f(x_1, x_2)$ is integrable with respect to x_1. When f is continuous, the corollary says that this is not necessary.*

Add Gunning definition of integral on Jordan measurable set...

Add COV formula (assuming Jordan measurable...)

Add spherical coordinates...

Add divergence theorem (and how differentials change)...

Improper integrals:

Add general limiting definition from Gunning...

Add discussion of moderate decrease from Stein-Shakarchi...

Add statement of COV...

Add spherical coordinates...

Interchange theorems:

Source: Lang, "Undergraduate Analysis", Chapters 9-2 + 13-3

Limits/integrals/derivatives of sequences:

Theorem 0.6. *If $\{f_n\}$ are continuous on $[a,b]$ and $\{f_n\}$ converges uniformly to f on $[a,b]$, then $\lim_{x\to x_0} f_n(x) = f(x_0)$.*

Proof. Clear from the fact that the uniform limit f must be continuous. □

Theorem 0.7. *If $\{f_n\}$ are continuous on $[a,b]$ and $\{f_n\}$ converges uniformly to f on $[a,b]$, then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.*

Theorem 0.8. *Suppose that $\{f_n\}$ are C^1 on $[a,b]$, $\{f_n'\}$ converges uniformly to a function g, and there exists $x_0 \in [a,b]$ for which $\{f_n(x_0)\}$ converges. Then $\{f_n\}$ converges uniformly to a differentiable function f such that $f' = g$.*

Limits/derivatives of series:

These results are corollaries of the results on sequences, as a series is simply the limit of its partial sums.

Theorem 0.9. *If $\{f_n\}$ are continuous on $[a,b]$ and the series $\sum f_n$ converges uniformly to f, then $\lim_{x\to x_0} \sum f_n(x) = f(x_0)$.*

Theorem 0.10. *Suppose that f_n are C^1 on $[a,b]$, the series $\sum f_n'$ converges uniformly to a function g, and there exists x_0 for which $\sum f_n(x_0)$ converges. Then $\sum f_n$ converges uniformly to a differentiable function f such that $f' = \sum_n f_n'$.*

Limits of (improper) integrals:

Definition 0.11. *Let $\phi(y) = \int_a^\infty f(x,y)dx$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable. We say that $\phi(y)$ **converges uniformly** in y if the limit $\lim_{b\to\infty} \int_a^b f(x,y)dx$ converges uniformly in y.*

Remark 0.12. *Note that the definition may be extended to all other cases when a or b is infinite.*

Theorem 0.13. *Let $\phi(y) = \int_a^b f(x,y)dx$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable. (Here, a, b are allowed to be $\pm\infty$.) Suppose that f is continuous on $[a,b] \times \mathbb{R}$ and $\int_a^b f(x,y)dx$ converges uniformly in y. Then for all $y \in \mathbb{R}$, $\lim_{y\to y_0} \phi(y) = \int_a^b f(x,y_0)$, i.e. $\phi(y)$ is continuous on $\mathbb{R}$.*

Proof. Sketch: write out the difference. When the integral is improper, split the integral into a bounded interval and tails. Use uniform continuity for the interval and uniform convergence for the tails. □

Remark 0.14. *If a, b are finite, we do not need uniform convergence. It suffices to check that f is continuous on $(x,y) \in [a,b] \times \mathbb{R}$. Further, if we only need to show the result for some $y_0 \in \mathbb{R}$, it is sufficient that f is continuous on a compact neighborhood of y_0.*

Derivatives of (improper) integrals (Leibniz rule):

Sources: Lang (above) and Conrad, `https://kconrad.math.uconn.edu/blurbs/analysis/diffunderint.pdf`

Theorem 0.15 (Leibniz rule)**.** *Let $\phi(y) = \int_a^b f(x,y)dx$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable. (Here, a, b are allowed to be $\pm\infty$.) Suppose that:*

1. *f and $\partial_y f$ are continuous on $[a,b] \times \mathbb{R}$;*
2. *there exist $A(x), B(x)$ s.t. $|f(x,y)| \leqslant A(x)$, $|\partial_y f(x,y)| \leqslant B(x)$ with $\int_a^b A(x)$ and $\int_a^b B(x)$ convergent.*

Then $\phi(y)$ is differentiable and $\partial_y \phi(y) = \int_a^b \partial_y f(x,y)dx$.

Proof. Sketch: write out the difference quotient. When the integral is improper, split the integral into a bounded interval and tails. Use 1 for the interval and 2 for the tails. □

Remark 0.16. *If a, b are finite, we do not need condition 1. It suffices to check that $f, \partial_y f$ are continuous on $(x,y) \in [a,b] \times \mathbb{R}$. Further, if we only need to show the result for some $y_0 \in \mathbb{R}$, it is sufficient that $f, \partial_y f$ are continuous on a compact neighborhood of y_0.*

Remark 0.17. *Condition 2 may be regarded as "uniform convergence" of $\int f, \int \partial_y f$ in the variable y. In this light, we see the connection to interchange of derivatives and series of functions, for which uniform convergence is also required. (Note: Lang specifically refers to condition 2 as "absolute convergence".)*